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A new method to find homoclinic and heteroclinic orbits

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ABSTRACT

A new series method is provided for continuous-time autonomous dynamical systems, which can find exact orbits as opposed to approximate ones. The method can reduce the connecting orbit problem as a boundary value problem in an infinite time domain to the initial value problem. It consists of transforming time to the logarithmic scale, substituting a power series around each fixed point of interest for each of the unknown functions into the system, and equating the corresponding coefficients. When solving for the power series coefficients, additional parameters are used in order to find the intersections of the unstable manifold and the stable manifold of the equilibria. This paper demonstrates how the new method allows to obtain heteroclinic and homoclinic orbits in some well-known cases, such as Nagumo system, stretch-twist-fold flow or mathematical pendulum.

1. Introduction

The research of dynamical systems has attracted attention of many scholars during the past decades. Nevertheless, it still remains an active field today, with many interesting open questions. From a dynamical system point of view, we are greatly interested in the long-term behavior of solutions of differential equations. In particular, it is exciting how to find homoclinic or heteroclinic orbits. Homoclinic orbits often arise as the limiting case of periodic solutions, while heteroclinic ones represent traveling wave solutions of parabolic partial differential equations [1]. More importantly, the existence of homoclinic or heteroclinic orbits is critical for applying the Šilnikov theorem, which provides a very useful tool for proving chaos in continuous-time autonomous systems [2–4].

Though we can find homoclinic or heteroclinic orbits by the Hamilton function for a Hamiltonian system, it is very difficult to find these orbits for a system that is not Hamiltonian, especially as a function of time t. In more recent years, a great deal of research has been invested in the proof of existence and computation of homoclinic or heteroclinic orbits [5–16]. In order to obtain numerical or analytical solutions, the methods include numerical computation [1,8–11] and applying series [12–16]. The former was developed into many variational methods, such as arclength parameterization method [1] and Hermite spectral method [11]. A series method was proposed by Zhou et al. to find exact orbits [12]. The method has since been applied and improved [14–16].

In this paper, we introduce a new series method for continuous-time autonomous dynamical systems, which can find exact orbits as opposed to approximate ones. The basis of the approach is to find the intersections of the unstable manifold and the stable manifold of the equilibria. By using the logarithmic transformation of time, the connecting orbit problem as a boundary value problem in an infinite time domain is reduced to the initial value problem. The solutions are formally expressed as power series expansions, and the coefficients are determined in virtue of the undetermined coefficient method. Finally, the method is successfully applied to find the homoclinic or heteroclinic orbits in some systems.

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This paper is organized as follows. Section 2 investigates the new series method in detail. Section 3 describes how to apply the new method to find the exact heteroclinic orbits in the stretch-twist-fold flow. Section 4 describes how to apply the method to find the exact heteroclinic orbit in the Nagumo system. Section 5 illustrates how to deal with the system whose right-hand side is of the non-polynomial form. We also discuss how to deal with homoclinic orbits in Section 6.

2. Scaling logarithm change series method

Consider the following system

$$\frac{dx}{dt} = f(x, \lambda), \quad t \in R, \tag{2.1}$$

where $x \in R^n$, $\lambda \in R^m$, and $f : R^n \times R^m \to R^n$ is sufficiently smooth. For a given λ, if there exists a non-constant solution $x = x(t, \lambda)$ of (2.1) such that

$$P_+ = \lim_{t\to+\infty} x(t,\lambda), \quad P_- = \lim_{t\to-\infty} x(t,\lambda), \quad f(P_\pm, \lambda) = 0,$$

then $x = x(t, \lambda)$ is called a connecting orbit between the equilibria P_+ and P_-. If $P_+ = P_-$, the orbit $x = x(t, \lambda)$ is called a homoclinic orbit; otherwise, it is called a heteroclinic orbit. $W^u_+(W^u_-)$ denotes the unstable manifold of $P_+(P_-)$, and $W^s_+(W^s_-)$ denotes the stable manifold of P_+ (P_-), respectively.

The paper provides a new series method, named scaling logarithm change series (SLS) method, to find homoclinic or heteroclinic orbits in continuous-time autonomous dynamical systems. The basis of the approach is to find the intersections of the unstable manifold W^u_- and the stable manifold W^s_+ of the equilibria. The main steps are described as follows. For simplicity, we will mainly illustrate how to find homoclinic or heteroclinic orbits for polynomial systems. However, as shown in Section 5, the method is also applicable to non-polynomial analytic systems.

Step 1. For $t > 0$, introduce the following logarithmic scale in (2.1):

$$t = -\frac{1}{T_1}\ln(\tau), \tag{2.2}$$

where T_1 is an undetermined positive real constant, called *scaling factor*. Obviously, $\tau \to +0$ as $t \to +\infty$. Hence, $t > 0$ is transformed into $0 < \tau < 1$.

Using the transformation (2.2) and (2.1) becomes

$$-T_1\tau\frac{dx}{d\tau} = f(x). \tag{2.3}$$

Step 2. We assume that system (2.3) has a solution of the form

$$x_i(\tau) = a_0^{(i)} + \sum_{k=1}^{\infty} a_k^{(i)}\tau^k \quad (i = 1, 2, \ldots, n), \tag{2.4}$$

where $\left(a_0^{(1)}, a_0^{(2)}, \ldots, a_0^{(n)}\right) = P_+$, and $a_k^{(i)}$ $(k \geqslant 1, i = 1, 2, \ldots, n)$ are undetermined coefficients.

Substituting (2.4) into (2.3) gives

$$-T_1\tau\sum_{k=1}^{\infty} k a_k^{(i)}\tau^{k-1} = f_i\left(a_0^{(1)} + \sum_{k=1}^{\infty} a_k^{(1)}\tau^k, a_0^{(2)} + \sum_{k=1}^{\infty} a_k^{(2)}\tau^k, \ldots, a_0^{(n)} + \sum_{k=1}^{\infty} a_k^{(n)}\tau^k\right), \quad i = 1, 2, \ldots, n. \tag{2.5}$$

Step 3. Comparing the coefficients of τ^1 in (2.5), we have

$$(T_1 I + J(P_+))\begin{pmatrix} a_1^{(1)} \\ a_1^{(2)} \\ \vdots \\ a_1^{(n)} \end{pmatrix} = 0, \tag{2.6}$$

where $J(P_+)$ is the Jacobian of system (2.1) evaluated at the equilibrium P_+.