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Function-valued Padé-type approximant via E-algorithm and its applications in solving integral equations ☆

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ABSTRACT

The function-valued Padé-type approximant (*FPTA*) was defined in the inner product space [8]. In this work, we choose the coefficients in the Neumann power series to make the inner product with both sides a function-valued system of equations to yield a scalar system. Then we express an *FPTA* in the determinant form. To avoid the direct computation of the determinants, we present the E-algorithm for *FPTA* based on the vector-valued E-algorithm given by Brezinski [4]. The method of *FPTA* via E-algorithm (*FPTAVEA*) not only includes all previous methods but overcomes their essential difficulties. The numerical experiment for a typical integral equation [1] illustrates that the method of *FPTAVEA* is simpler and more effective for obtaining the characteristic values and the characteristic functions than all previous methods. In addition, this method is also applicable to other Fredholm integral equations of the second kind without explicit characteristic values and characteristic functions. A corresponding example [12] is given and the numerical result is the same as that in [12].

1. Introduction

Consider a Fredholm integral equation of the second kind

$$f(x,\lambda) = g(x) + \lambda \int_a^b K(x,y) f(y,\lambda)dy, (x,y) \subset [a,b] \times [a,b], \tag{1}$$

where $g(x) \in L_2[a,b]$ and $K(x,y)$ defined in $[a,b] \times [a,b]$ is an L_2 kernel. The technique utilized for solving the integral equation is based on successive substitution, which is an iterative procedure, yielding a sequence of approximations leading to an infinite power series solution. We now consider the generating function $f(x,\lambda)$ in λ of this series. Let $f(x,\lambda)$ be analytic at the origin $\lambda = 0$ and meromorphic in a neighborhood of $\lambda = 0$, and let its Neumann series be given by

$$f(x,\lambda) = \sum_{i=0}^{\infty} c_i(x)\lambda^i, \tag{2}$$

where $c_i(x) \in L_2[a,b]$ and $[a,b]$ is the domain of definition of $c_i(x)$ in some natural sense, and

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$$c_i(x) = \int_a^b K(x,y)c_{i-1}(y)dy \quad \text{for } i \geqslant 1, \tag{3}$$

with $c_0(x) = g(x)$.

Definiton 1. [[12]] A characteristic value of the integral Eq. (1) is a scalar λ such that

$$f(x,\lambda) = \lambda \int_a^b K(x,y)f(y,\lambda)dy$$

possesses a non-trivial solution and the solution of (1) corresponding to the characteristic value is a characteristic function.

Our goal in this paper is to compute the characteristic values of the integral Eq. (1). There are many methods for the estimation of the characteristic values and characteristic functions of (1) such as the classic Padé approximant (*CPA*) [5,6], the generalized inverse function-valued Padé approximant (*GIPA*) [5,6], the modified Padé approximant (*MPA*) [10], the square Padé approximant (*SPA*) [10], the integral Padé approximant (*IPA*) [10], the function-valued Padé-type approximant (*FPTA*) [7] and the ε-algorithm of function-valued *GIPA* [7,9]. Gu and Shen introduced a kind of function-valued Padé approximant via the formal orthogonal polynomials (*FPTAVOP*) [8]. This method gave a better approximation to characteristic values and characteristic functions of (1) than the methods mentioned above.

In this work, following the definition of *FPTA*, we choose the $c_k(x)$, $k = 0, 1, \ldots$ to make the inner product in $L_2[a,b]$ with both sides a function-valued system of equations to yield a scalar system from which we present the determinantal expression for *FPTA*. To avoid the direct computation of the determinant, we give the *E*-algorithm for *FPTA*(*FPTAVEA*) based on the vector-valued *E*-algorithm presented by Brezinski [4]. The method of *FPTAVEA* not only includes all previous methods but also overcomes their essential difficulties. The Numerical experiment for a typical integral equation [1] illustrates that the method of *FPTAVEA* is simpler and more effective for obtaining the characteristic values and the characteristic functions than all existing methods. In addition, the method of *FPTAVEA* is also applicable to other Fredholm integral equations of the second kind without explicit characteristic values and characteristic functions. A corresponding example[12] is given and the numerical result is the same as that in [12].

2. Function-valued Padé-type approximant (*FPTA*)

In this section, following the concept of the scalar Padé-type approximants [2,3], we redefine *FPTA* referred in [8].

Let $f(x,\lambda)$ be the Numann series in λ in the form (2). Let $c^{(l)} : \mathbf{P} \to \mathbb{C}$ be a linear functional on the polynomial space $\mathbf{P}$:

$$c^{(l)}(t^i) = c_{l+i}(x), \quad i = 0, 1, \ldots, l \in \mathbb{Z}, \tag{4}$$

where $c^{(l)}(t^i) = 0$ for $l + i < 0$.

Let $|t\lambda| < 1$. From the linear functional $c = c^{(0)}$ in (4), it follows that

$$c((1-t\lambda)^{-1}) = c(1 + t\lambda + (t\lambda)^2 + \cdots) = \sum_{i=0}^{\infty} c_i(x)\lambda^i = f(x,\lambda).$$

Let $v(\lambda)$ be a scalar polynomial of degree n defined in $\mathbf{P}_n$, that is,

$$v_n(\lambda) = b_0 + b_1\lambda + \cdots + b_n\lambda^n, \tag{5}$$

where $b_n \neq 0$.

We define a function-valued polynomial $w_m(x,\lambda)$ in λ by

$$w_m(x,\lambda) = c\left(\frac{t^{m-n+1}v_n(t) - \lambda^{m-n+1}v_n(\lambda)}{t-\lambda}\right). \tag{6}$$

It is clear that $w_m(x,\lambda)$ is of degree m in λ.

Let us define

$$\tilde{v}_n(\lambda) = \lambda^n v_n(\lambda^{-1}) = \sum_{j=0}^{n} b_j\lambda^{n-j}, \tag{7}$$

$$\tilde{w}_m(x,\lambda) = \lambda^m w_m(x,\lambda^{-1}). \tag{8}$$

It is found from (7) that $\tilde{v}_n(0) \neq 0$ implies $b_n \neq 0$.

From (6)–(8), $\tilde{w}(x,\lambda)$ is of the following form:

$$\tilde{w}_m(x,\lambda) = \sum_{j=0}^{n} b_j\lambda^{n-j} \sum_{i=0}^{m-n+j} c_i(x)\lambda^i = \sum_{i=0}^{m}\sum_{j=0}^{n} b_j\tilde{c}_i^j(x)\lambda^i, \tag{9}$$

where $\tilde{c}_i^j(x) = c_{i-n+j}(x)$.