Applied Mathematics and Computation 217 (2011) 6838–6855

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Optimal control of swinging alliances in a parabolic competition model

E. Asano [a,*], S. Lenhart [b,c], V. Protopopescu [c]

[a] Department of Environmental Science, Policy and Geography University of South Florida, St. Petersburg, FL 33701, USA
[b] Department of Mathematics, University of Tennessee, Knoxville, TN 37996-0612, USA
[c] Computational Sciences and Engineering Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831-6016, USA

ARTICLE INFO

Keywords:
Optimal control
Parabolic PDEs
Competition
Cooperation
Opportunistic alliances

ABSTRACT

A system of parabolic partial differential equations describes the interaction of three populations, modeling a dynamic competition/cooperation scenario. More precisely, two populations are always competing with each other, but the third population can switch the mode of alliance with the other two populations between cooperation and competition. The control is a function measuring the strength and nature of the alliance and the goal is to maximize the population with the swinging alliance while keeping the other two populations close to each other and minimizing the cost of the alliance action. Various scenarios are illustrated with numerical results.

1. Introduction

Balancing the interests of the two competing populations with an alliance involving the third population is modeled by a parabolic system. We consider optimal control of a nonlinear system of parabolic partial differential equations with Dirichlet boundary conditions in a bounded, space–time domain $Q = \Omega \times (0,T)$, $\Omega \subset R^n$. Solutions of the system represent populations of three species. One of the populations can switch its alliance between cooperation and competition with the other two populations. The other two populations always compete with each other. The control is the function α, measuring the strength of interaction; the sign of α tells whether the interaction is competitive or cooperative. The control set is defined as

$$U \equiv \{\alpha \in L^\infty(Q) : |\alpha(x,t)| \leqslant \overline{M} \quad \text{a.e. in } Q\},$$

where $\overline{M} > 0$. Given a control $\alpha \in U$, the corresponding state variables, $u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ satisfy the state system:

$$L_k u_k = F_k(u_1, u_2, u_3, \alpha) + f_k, \quad \text{for } k = 1,2,3. \tag{1}$$

ICs:

$$u_k(x,0) = u_{k0}(x) \quad \text{for } x \in \Omega,\ k = 1,2,3. \tag{2}$$

BCs:

$$u_k = 0 \quad \text{on } \partial\Omega \times (0,T),\ k = 1,2,3, \tag{3}$$

where

$$L_k u_k \equiv (u_k)_t - \sum_{i,j=1}^n \left(a_{ij}^k (u_k)_{x_i}\right)_{x_j} + \sum_{i=1}^n (b_i)^k (u_k)_{x_i} + c^k u_k \tag{4}$$

* Corresponding author.
E-mail addresses: easano@mail.usf.edu (E. Asano), lenhart@math.utk.edu (S. Lenhart).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2011.01.029

and

$$\begin{aligned} F_1(u_1,u_2,u_3,\alpha) &= -u_1 \int_\Omega \frac{u_2}{1+u_2}dx + \alpha u_1 \int_\Omega \frac{u_3}{1+u_3}dx, \\ F_2(u_1,u_2,u_3,\alpha) &= -u_2 \int_\Omega \frac{u_1}{1+u_1}dx - \alpha u_2 \int_\Omega \frac{u_3}{1+u_3}dx, \\ F_3(u_1,u_2,u_3,\alpha) &= \alpha u_3 \int_\Omega \frac{u_1}{1+u_1}dx - \alpha u_3 \int_\Omega \frac{u_2}{1+u_2}dx. \end{aligned} \tag{5}$$

The autonomous sources $f_i's$ represent either immigration or emigration. The first and second terms on the right hand of each equation represent the non-local interaction between two populations. For example, the terms in the right hand side of the u_1 PDE represent the non-local interaction between populations 1 and 2 and populations 1 and 3, respectively. The main reason why the interaction terms have the form, $\int_\Omega \frac{u_i}{1+u_i}dx$, is to bound the states. Indeed, if the interaction term were simply of the form, $\int_\Omega u_i dx$, and the source terms are positive then the solutions may blow up at finite time, since quadratic growth terms may cause such behavior. Interactions of this form have been considered in combat modeling involving coalitions [10].

The objective functional, defined from the perspective of the opportunistic population 3 is:

$$J(\alpha) = \frac{1}{2}\int_Q \left[Ku_3^2 - L(u_2-u_1)^2 - M\alpha^2\right]dx\,dt, \tag{6}$$

where K, L and M are positive weighting constants, which balance the importance of the three terms. The second term in the integrand reflects the potential risk incurred by population 3 from the disparity between populations 1 and 2. The last term reflects the cost of switching allegiance. The goal is to maximize the size of population 3 while keeping the sizes of the other two populations close to each other and minimizing cost.

We seek to maximize the functional over the admissible class of control space such that

$$J(\alpha^*) = \max_{\alpha\in U} J(\alpha). \tag{7}$$

For work on related problems with coalitions and competitions, see [1,9,8,6,4,15,10]. For background on control of PDEs, see the fundamental book by Lions [13] and the book by Li and Yong [12].

In Section 2, we prove the existence of solutions to the state system and *a priori* estimates for the state solutions. In Section 3, we prove the existence of an optimal control. The control-to-state map is differentiated to obtain the sensitivity system. Using the sensitivity system, we derive the optimality system by differentiating the objective functional with respect to the control in the fourth section. The optimal control is explicitly expressed in terms of the solutions to the optimality system, which consists of the state system coupled with an adjoint system. In Section 5, we prove the uniqueness of the optimal control. Finally, we show numerical results using some simple examples, with spatially independent control functions, $\alpha(t)$, and with more general control functions, $\alpha(x,t)$, respectively.

2. Assumptions

We make the following assumptions:

$$u_{k0}(x) \in L^\infty(\Omega), \quad \text{for } k = 1,2,3, \tag{8}$$

$$a_{ij}^k \in C^1(\overline{Q}), \quad a_{ij}^k = a_{ji}^k \quad \text{for } k=1,2,3, \quad i,j = 1,2,3,\dots,n, \tag{9}$$

$$b_i^k \in C^1(\overline{Q}), \quad c_i^k \in C(\overline{Q}) \quad \text{for} \quad k=1,2,3, \quad i=1,2,3,\dots,n, \tag{10}$$

$$\sum_{i,j=1}^n a_{ij}^k(x,t)\xi_i\xi_j \geqslant \theta\xi_i^2 \quad \text{for } k=1,2,3, \quad \text{where } \theta>0, \text{ for all } (x,t)\in Q, \xi\in R^n, \tag{11}$$

$$f_k \in L^\infty(Q), \quad \text{and } f_k(x,t) \geqslant 0 \quad \text{for all } (x,t)\in Q \quad \text{for } k=1,2,3. \tag{12}$$

The underlying state space for system (1)–(5) is $V = L^2(0,T;H_0^1(\Omega))$.

Definition 1. For each $t \in (0,T)$, we define the bilinear form in $H^1(\Omega)$:

$$a^k(t,\psi,\phi) = \int_\Omega \sum_{i,j=1}^n a_{ij}^k \psi_{x_i}\phi_{x_j}dx + \int_\Omega \sum_{i=1}^n (b_i)^k \psi_{x_i}\phi dx + \int_\Omega c^k\psi\phi dx$$

for k = 1, 2, 3.

Definition 2. (u_1, u_2, u_3) in V^3 is a solution of system (1)–(5) provided

(i) $(u_1)_t, (u_2)_t, (u_3)_t \in L^2(0,T;H^{-1}(\Omega))$,

(ii) $\int_0^T(\langle(u_k)_t,\phi_k\rangle + a^k(t,u_k,\phi_k))dt = \int_Q(F_k(u_1,u_2,u_3,\alpha)+f_k)\phi_k dx\,dt$,
for all $\phi_k \in L^2(0,T;H_0^1(\Omega))$ and $k=1,2,3$,

(iii) $u_1(x,0) = u_{10}(x), \quad u_2(x,0) = u_{20}(x), \quad u_3(x,0) = u_{30}(x) \quad$ for $x\in\Omega$.

where $\langle(u_k)_t,\phi_k\rangle$ denotes duality action between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.