



Positive solutions for boundary value problems of nonlinear fractional differential equations ☆

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ABSTRACT

In this paper, we study the existence of positive solutions for the nonlinear fractional differential equation boundary value problem.

$$\begin{aligned} D_{0+}^{\alpha} u(t) &= \lambda f(u(t)), \quad 0 < t < 1, \\ u(0) + u'(0) &= 0, \quad u(1) + u'(1) = 0, \end{aligned}$$

where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} is the Caputo fractional derivative, $\lambda > 0$ and $f: [0, +\infty) \rightarrow (0, +\infty)$ is continuous. By using the properties of the Green function and Guo–Krasnosel'skii fixed point theorem on cones, the eigenvalue intervals of the nonlinear fractional differential equation boundary value problem are considered, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problem are established.

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1. Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications, see [1–6]. It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations on terms of special functions. Recently, there are some papers dealing with the existence of solutions (or positive solutions) of nonlinear initial fractional differential equation by the use of techniques of nonlinear analysis (fixed-point theorems, Leray–Schauder theory, Adomian decomposition method, etc.), see [7–10]. In fact, there has the same requirements for boundary conditions. However, there are few papers considered the boundary value problems of fractional differential equation, see [11–16].

In [12], Zhang considered the existence of positive solutions for the following problem

$$\begin{aligned} D_{0+}^{\alpha} u(t) &= f(t, u(t)), \quad 0 < t < 1, \\ u(0) + u'(0) &= 0, \quad u(1) + u'(1) = 0, \end{aligned}$$

where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} is the Caputo fractional derivative, and $f: [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous. By using the properties of the Green function and fixed point theorem on cones, some existence and multiplicity results of positive solutions were obtained.

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To the best of our knowledge, there is very little known about the existence of positive solutions for the following problem

$$D_{0+}^{\alpha} u(t) = \lambda f(u(t)), \quad 0 < t < 1, \quad (1.1)$$

$$u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0, \quad (1.2)$$

where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} is the Caputo fractional derivative, $\lambda > 0$ and $f: [0, +\infty) \rightarrow (0, +\infty)$ is continuous.

In this paper, analogy with boundary value problems for differential equations of integer order, we firstly give the corresponding Green function named by fractional Green's function and some properties of the Green function, see [12]. Consequently, the problem (1.1), (1.2) is reduced to a equivalent Fredholm integral equation. Finally, by using the properties of the Green function and Guo–Krasnosel'skii fixed point theorem on cones, the eigenvalue intervals of the nonlinear fractional differential equation boundary value problem are considered, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problem are established.

2. Preliminaries

For the convenience of the reader, we give some background materials from fractional calculus theory to facilitate analysis of problem (1.1), (1.2). These materials can be found in the recent literature, see [3,12,17,18].

Definition 2.1. ([17]) The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f: (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where n is the smallest integer greater than or equal to α , provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2. ([17]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f: (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right side is pointwise defined on $(0, +\infty)$.

Remark 2.1. ([3]) By Definition 2.1, under natural conditions on the function $f(t)$, for $\alpha \rightarrow n$ Caputo's derivative becomes a conventional n -th derivative of the function $f(t)$.

Remark 2.2. As a basic example,

$$D_{0+}^{\alpha} t^{\mu} = \mu(\mu-1) \cdots (\mu-n+1) \frac{\Gamma(1+\mu-n)}{\Gamma(1+\mu-\alpha)} t^{\mu-\alpha}, \quad \text{for } t \in (0, \infty),$$

given in particular $D_{0+}^{\alpha} t^{\mu} = 0$, $\mu = 0, 1, \dots, n-1$, where D_{0+}^{α} is the Caputo fractional derivative, n is the smallest integer greater than or equal to α .

In fact, for $t \in (0, +\infty)$,

$$\begin{aligned} D_{0+}^{\alpha} t^{\mu} &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{(t^{\mu})^{(n)}}{(t-s)^{\alpha-n+1}} ds = \frac{1}{\Gamma(n-\alpha)} \mu(\mu-1) \cdots (\mu-n+1) t^{\mu-\alpha} \int_0^1 z^{\mu-n} (1-z)^{n-\alpha-1} dz \\ &= \mu(\mu-1) \cdots (\mu-n+1) \frac{\Gamma(1+\mu-n)}{\Gamma(1+\mu-\alpha)} t^{\mu-\alpha}. \end{aligned}$$

Thus,

$$D_{0+}^{\alpha} t^{\mu} = \mu(\mu-1) \cdots (\mu-n+1) \frac{\Gamma(1+\mu-n)}{\Gamma(1+\mu-\alpha)} t^{\mu-\alpha} = 0, \quad \text{for } \mu = 0, 1, \dots, n-1.$$

From the definition of the Caputo derivative and Remark 2.2, we can obtain the following statement.

Lemma 2.1. Let $\alpha > 0$. Then the fractional differential equation

$$D_{0+}^{\alpha} u(t) = 0$$

has $u(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1}$, $C_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, as unique solutions, where n is the smallest integer greater than or equal to α .

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