Applied Mathematics and Computation 217 (2011) 6974–6980

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the eigenvalue estimation for solution to Lyapunov equation ☆

Yang Xingdong [a,*], Ding Zhiying [b], Zhang Jiajing [a], Sun Suya [a]

[a] Department of Mathematics, Nanjing University of Information Science and Technology, Nanjing 210044, China
[b] College of Atmospheric, Nanjing University of Information Science and Technology, Nanjing 210044, China

ARTICLE INFO

Keywords:
Lyapunov equation
Eigenvalue
Singular value
Characteristic estimation

ABSTRACT

This paper deals with the problems of eigenvalue estimation for the solution to the perturbed matrix Lyapunov equation. We obtain some eigenvalue inequalities on condition that X is a positive semidefinite solution to the equation $A^TXA - X = -Q$, which can be used in control theory and linear system stability.

1. Introduction

In this paper, let $C^{m\times n}$ be the set of $m\times n$ complex matrices, and $C_r^{m\times n}$, consisting of matrices with rank r, be the subset of $C^{m\times n}$. Let I_r be the identity matrix of order r. Given $A\in C_r^{m\times n}$. The symbols A^T, A^H, and $r(A)$ stand for the transpose, conjugate transpose, and rank of A, respectively. Let $A\in C^{m\times n}$. Denote the eigenvalues of A by $\lambda_1(A),\lambda_2(A),\ldots,\lambda_n(A)$, and the singular values of A by $\sigma_1(A),\sigma_2(A),\ldots,\sigma_n(A)$. We further assume that the eigenvalues, if they are all real, and the singular values are arranged in decreasing order. For Hermitian matrices A and B, as usual, we write $A\geqslant 0$ if A is positive semidefinite (nonnegative definite), $A>0$ if $A\geqslant 0$ and A is nonsingular, and $A\geqslant B$ if $A-B\geqslant 0$.

We then turn our attention to investigate the discrete Lyapunov matrix equation

$$A^TXA - X = -Q, \tag{1}$$

where $A\in R^{n\times n}$, and $Q\geqslant 0$.

Such a type of the equation arouses many applications in control theory and linear system stability [1–9]. If the coefficient matrix A in (1) is perturbed, then the equality (1) can be characterized by the following equation

$$X = Q + (A+\Delta A)^TX(A+\Delta A), \tag{2}$$

When both matrices A and Q in (1) are perturbed, the equality (1) can be written as

$$X = Q + \Delta Q + (A+\Delta A)^TX(A+\Delta A), \tag{3}$$

where $Q\geqslant 0$, and $\Delta Q\geqslant 0$.

There is a unique symmetric positive definite solution to Eqs. (2) and (3)—assuming that both the matrix pares $\left(A+\Delta A,(Q+\Delta Q)^{\frac{1}{2}}\right)$ and $\left(A+\Delta A,Q^{\frac{1}{2}}\right)$ are steady [10]. In the following Theorems 1–4, we always assume that the matrix pare in (2) or (3) is steady.

☆ The first author's work is supported by National Natural Science Foundation of China (No. 40975037).
* Corresponding author.
E-mail address: xingdongy@hotmail.com (X. Yang).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2011.01.106

Lemma 1 (*LidskiWielandt, [11]*). *Let A and $B \in C^{n\times n}$ be Hermitian matrix. Then*

$$\sum_{t=1}^{k}\lambda_{i_t}(A+B) \geqslant \sum_{t=1}^{k}\lambda_{i_t}(A) + \sum_{t=1}^{k}\lambda_{n-t+1}(B), \tag{4}$$

$$\sum_{t=1}^{k}\lambda_{i_t}(A+B) \leqslant \sum_{t=1}^{k}\lambda_{i_t}(A) + \sum_{t=1}^{k}\lambda_{t}(B), \tag{5}$$

where $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$.

In particular, when $i_t = 1,2,\ldots,k$, both inequalities (4) and (5) can be respectively changed into the following forms:

$$\sum_{i=1}^{k}\lambda_{i}(A+B) \geqslant \sum_{i=1}^{k}\lambda_{i}(A) + \sum_{i=1}^{k}\lambda_{n-i+1}(B),$$

$$\sum_{i=1}^{k}\lambda_{i}(A+B) \leqslant \sum_{i=1}^{k}\lambda_{i}(A) + \sum_{i=1}^{k}\lambda_{i}(B).$$

Lemma 2 [12]. *Let A and $B \in C^{n\times n}$ be positive semidefinite Hermite matrix. Then*

$$\sum_{t=1}^{k}\lambda_{i_t}(A)\lambda_{n-i_t+1}(B) \leqslant \sum_{t=1}^{k}\lambda_{i_t}(AB) \leqslant \sum_{t=1}^{k}\lambda_{i_t}(A)\lambda_{t}(B),$$

where $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$.

Lemma 3. *Let $A, B \in H^{n\times n}$. Then*

(i) $$\sum_{t=1}^{k}\lambda_{n-t+1}(A+B) \leqslant \sum_{t=1}^{k}\lambda_{i_t}(A) + \sum_{t=1}^{k}\lambda_{n-i_t+1}(B).$$

(ii) $$\sum_{t=1}^{k}\lambda_{t}(A+B) \geqslant \sum_{t=1}^{k}\lambda_{i_t}(A) + \sum_{t=1}^{k}\lambda_{n-i_t+1}(B).$$

(iii) $$\sum_{t=1}^{k}\lambda_{n-i_t+1}(A+B) \geqslant \sum_{t=1}^{k}\lambda_{n-t+1}(A) + \sum_{t=1}^{k}\lambda_{n-i_t+1}(B).$$

where $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$.

Proof. By the inequality (4) in Lemma 1, we get

$$\sum_{t=1}^{k}\lambda_{i_t}(A) = \sum_{t=1}^{k}\lambda_{i_t}[(-B)+(A+B)] \geqslant \sum_{t=1}^{k}\lambda_{i_t}(-B) + \sum_{t=1}^{k}\lambda_{n-t+1}(A+B) = -\sum_{t=1}^{k}\lambda_{n-i_t+1}(B) + \sum_{t=1}^{k}\lambda_{n-t+1}(A+B).$$

This implies the inequality (i) holds. Similarly, we can prove the inequalities (ii). By using inequality (i), we can infer that

$$\sum_{t=1}^{k}\lambda_{n-t+1}(A) \leqslant \sum_{t=1}^{k}\lambda_{n-i_t+1}(A+B) + \sum_{t=1}^{k}\lambda_{i_t}(-B) = \sum_{t=1}^{k}\lambda_{n-i_t+1}(A+B) - \sum_{t=1}^{k}\lambda_{n-i_t+1}(B).$$

Hence, inequality (iii) holds. □

2. Main results

Firstly, we investigate the matrix Eq. (2), in which the coefficient matrix A is perturbed only.

Theorem 1. *For any n-by-n real matrix A, perturbed matrix ΔA, and real positive semidefinite matrix Q, if X is a positive semidefinite solution to the perturbed matrix* Eq. (2), *and* $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant l \leqslant n$, *then*

$$\sum_{t=1}^{k}\lambda_{n-l+i_t}(X) \leqslant \sum_{t=1}^{k}\lambda_{t}(Q) + \sum_{t=1}^{k}\lambda_{n-l+i_t}(X)\sigma_{n-l+t}^{2}(A+\Delta A).$$