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Generalized reflective function and periodic solution of differential systems with small parameter

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ABSTRACT

In this paper, we study the generalized reflective matrix which can be represented by three exponential matrices and apply the results to discussing the existence of periodic solutions of systems with small parameter. The obtained results extend and improve the related conclusions of Musafirov.

1. Introduction

As is well known, to study the properties of differential system

$$x' = X(t,x) \tag{1}$$

is not easy. When it is 2ω-periodic with respect to t, i.e., $X(t+2\omega,x) = X(t,x)$, (ω is a positive constant), to study the behavior of solutions of (1), we could use, as introduced in [1], the Poincaré mapping. But it is very difficult to seek the Poincaré mapping for many systems which are not integrable in finite terms. In the 1980s, the Russian mathematician, Mironenko [2,3], first established the theory of reflective function (**RF**). Since then, a quite new method to study system (1) has been set. If $F(t,x)$ is **RF** of (1), then its Poincaré mapping can be expressed by: $T(x) = F(-\omega,x)$. So now we only need to seek RF. The literatures [4–11] are devoted to investigations of qualitative behavior of solutions of differential systems with help of reflective functions.

With the deepening of the study, the notion about reflective function has been extended in [7]. In the present section, we introduce the concept of the generalized reflective function(**GRF**), which will be used throughout the rest of this paper.

Now consider the system (1) with a continuously differentiable right-hand side and with a general solution $\varphi(t;t_0,x_0)$. For each such system, the **GRF** of (1) is defined as $F(t,x) = \varphi(\alpha(t);t,x)$, $(t,x) \in D \subset R \times R^n$, where $\alpha(t)$ is a continuously differentiable function such that $\alpha(\alpha(t)) = t$ and $\alpha(0) = 0$. Then, for any solution $x(t)$ of (1), we have $F(t,x(t)) = x(\alpha(t))$. Suppose system (1) is 2ω-periodic with respect to t, $F(t,x)$ is its **GRF**, if there exists a number η on R such that $\alpha(\eta) = 2\omega + \eta$, then $T(x) = F(\eta,x) = \varphi(\alpha(\eta);\eta,x)$ is the Poincaré mapping of (1) over the period $[\eta,\eta+2\omega]$. So, for any solution $x(t)$ of (1) defined on $[\eta,\eta+2\omega]$, it will be 2ω-periodic if and only if $F(\eta,x) = x$.

By the definition in [7], a continuously differentiable vector function $F(t,x)$ on $R \times R^n$ is called **GRF** if and only if it is a solution of the Cauchy problem

$$F_t(t,x) + F_x(t,x)X(t,x) = \alpha'(t)X(\alpha(t),F(t,x)); \quad F(0,x) = x. \tag{2}$$

The **GRF** of the linear system

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$$\dot{x} = P(t)x, \quad (t \in R, x \in R^n) \tag{3}$$

is also linear and $F(t,x) = F(t)x$, where the matrix $F(t)$ is called a generalized reflective matrix (**GRM**). This matrix is the solution of Cauchy problem (called a basic relation, **BR**)

$$F'(t) + F(t)P(t) = \alpha'(t)P(\alpha(t))F(t), \quad F(0) = E, \tag{4}$$

where E is the $n \times n$ unit matrix, $P(t)$ is a continuously differentiable $n \times n$ matrix function on R.

If $F(t)$ is the **GRM** of (3), then, it is also the **GRM** of the system

$$\dot{x} = P(t)x + F(\alpha(t))R(t)x + \alpha'(t)R(\alpha(t))F(t)x, \quad (t \in R, x \in R^n),$$

where $R(t)$ is an arbitrary continuously real matrix. So, we can apply the theory of **GRM** to discussing the properties of the solutions of such systems.

Suppose that $X(t)$ is the fundamental matrix of solutions of system (3), then the general solution of (3) is $x = \varphi(t;\tau,x) = X(t)X^{-1}(\tau)x$. Therefore, the **GRF** of (3) is $F(t) = X(\alpha(t))X^{-1}(t)$ and so $F(\alpha(t))F(t) \equiv F(0) = E$, where E is the $n \times n$ unit matrix.

As we known, when $P(t + 2\omega) = P(t)$, the fundamental matrix of (3) can be represented in the form $X(t) = \phi(t)e^{-Bt}$, where $\phi(t)$ is a continuous periodic $n \times n$ matrix and $\det \phi(t) \neq 0$, B is a constant $n \times n$ matrix, **GRM** of such systems is $F(t) = X(\alpha(t))X^{-1}(t) = \phi(\alpha(t))e^{(t-\alpha(t))B}\phi^{-1}(t)$.

With this in mind we assume that **GRM** of (3) is given by $F(t) = e^{A(\alpha(t))}e^{(t-\alpha(t))B}e^{-A(t)}$, where $A(t)$ is a continuously differentiable $n \times n$ matrix, B is a constant $n \times n$ matrix. Musafirov has discussed the form of **RM** $F(t) = e^{A(t)}e^{2Bt}e^{A(t)}$ and $F(t) = e^{\alpha(t)A}e^{\beta(t)B}e^{-\alpha(-t)A}$ in his papers [8,9]. In the following, we will study when the **GRM** of system (3) has the form of $F(t) = e^{A(\alpha(t))}e^{(t-\alpha(t))B}e^{-A(t)}$, where $\alpha(\alpha(t)) = t$ and $\alpha(0) = 0$. It will improve the conclusions of Musafirov [8,9].

2. Main results

In the following, we will always assume that $A(t)$ is a continuously differentiable $n \times n$ matrix with $A(t)A'(t) = A'(t)A(t)$, B is a constant $n \times n$ matrix, E is the $n \times n$ identity matrix.

Theorem 2.1

(1) *$F(t) = e^{A(\alpha(t))}e^{(t-\alpha(t))B}e^{-A(t)}$ is the* **GRM** *of system (3), if and only if*

$$F(t)D(t) - \alpha'(t)D(\alpha(t))F(t) = 0, \tag{5}$$

where $D(t) = P(t) - A'(t) - \alpha'(t)e^{A(t)}Be^{-A(t)}$.

(2) *Suppose that $\alpha'(0) = -1$ and $F(t) = e^{A(\alpha(t))}e^{(t-\alpha(t))B}e^{-A(t)}$ is the* **GRM** *of system (3), we have*

$$B = e^{-A(0)}(A'(0) - P(0))e^{A(0)} \tag{6}$$

and

$$\begin{aligned}
&(3\alpha''^2(0) + 2\alpha'''(0))(P(0) - A'(0)) + 2P''(0) - 2A'''(0) + 2A''(0)A'(0) - 2A'(0)A''(0) + 3\alpha''(0)P'(0) - 3\alpha''(0)A''(0)\\
&\quad + 4A'(0)P(0)A'(0) + P(0)[6A''(0) - 2A'^2(0) - 4P'(0) + 4A'(0)P(0) - 4P(0)A'(0) + 3\alpha''(0)A'(0) - 4\alpha''(0)P(0)]\\
&\quad + [4P'(0) - 6A''(0) - 2A'^2(0) - 4A'(0)P(0) + 4P(0)A'(0) + 4\alpha''(0)P(0) - 3\alpha''(0)A'(0)]P(0) = 0.
\end{aligned} \tag{7}$$

Proof

(1) According to the BR (4), we know that $F(t) = e^{A(\alpha(t))}e^{(t-\alpha(t))B}e^{-A(t)}$ is the **GRM** of system (3), if and only if

$$F(t)\left[P(t) - A'(t) - \alpha'(t)e^{A(t)}Be^{-A(t)}\right] - \alpha'(t)\left[P(\alpha(t)) - A'(\alpha(t)) - \alpha'(\alpha(t))e^{A(\alpha(t))}Be^{-A(\alpha(t))}\right]F(t) = 0,$$

i.e.

$$F(t)D(t) - \alpha'(t)D(\alpha(t))F(t) = 0,$$

where $D(t) = P(t) - A'(t) - \alpha'(t)e^{A(t)}Be^{-A(t)}$.

(2) By (1), we have the identity (5), putting $t = 0$, we get $D(0) = 0$ and

$$D(t) = P(t) - A'(t) - \alpha'(t)e^{A(t)}Be^{-A(t)}, \quad B = e^{-A(0)}(A'(0) - P(0))e^{A(0)}.$$

In view of $F'(0) = -2P(0)$, we twice differentiating the identity (5) and putting $t = 0$, then

$$2D''(0) - 4P(0)D'(0) + 4D'(0)P(0) + 3\alpha''(0)D'(0) = 0. \tag{8}$$