



A smooth approximation based on exponential spline solutions for nonlinear fourth order two point boundary value problems

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ABSTRACT

Methods of order two, four and six based on Exponential spline functions consisting of a polynomial part of degree three and an exponential part is developed to find approximation of linear and nonlinear fourth order two point boundary value problems. It is shown that the free parameter k of the exponential part can be used to raise the order of accuracy of the new scheme. Convergence analysis of these methods is shown along with numerical examples each for linear and nonlinear are included to illustrate the practical usefulness of our methods.

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1. Introduction

We will consider a smooth approximation of the solution of the following nonlinear two point boundary value problems:

$$y^{(4)} = f(x, y), \quad x \in [a, b] \quad (1.1)$$

subject to the boundary conditions:

$$y(a) - A_1 = y^{(1)}(a) - A_2 = 0, \quad y^{(2)}(b) - B_1 = y^{(3)}(b) - B_2 = 0. \quad (1.2)$$

For the linear problem $f(x, y) = g(x) - p(x)y$ where A_i and $B_i, i = 1, 2$ are finite real constants. Fourth order boundary value problems have attracted much attention in recent years. Problems of this type arise in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory; see for example [25,28]. It is assumed that $f(x, y)$ is real and continuous on $[a, b]$ and for the linear case, the functions $p(x)$ and $g(x)$ are continuous on $[a, b]$. The analytical solution of problem (1.1) and (1.2) cannot be obtained for arbitrary choices of $p(x)$ and $g(x)$, see [24]. For details of the existence and uniqueness of the problem (1.1) and (1.2), see [21,24].

The numerical analysis literature on differential equations contains little on the solution of the boundary value problems (1.1) and (1.2). Tien and Usmani [21] presented a survey on the numerical solution of the problem (1.1) and (1.2) using finite difference method, shooting method and quintic polynomial spline method. Also, Usmani [24], discussed a finite difference method and modified shooting technique based on fourth order Runge–Kutta method. On the other hand, the literature contains numerous methods formulated to find an approximate solution of the problem (1.1) with other types of boundary conditions involving first and second order derivatives. For example, Usmani [22–24], Rashidinia and Golbabaee [11] and Siddiqi and Akram [13,14] have developed and analyzed second order and fourth order convergent methods for the solution of linear fourth order two point boundary problem using quartic, quintic and sextic polynomial spline functions respectively. Al-Said et al. [1,2] demonstrated second order convergent method based on cubic and quartic polynomial spline functions for the

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solution of fourth order obstacle problems. Scott and Watts solved linear fourth order boundary value problem involving a parameter c in the following form:

$$y^{(4)} - (1+c)y^{(2)} + cy = -1 + \frac{1}{2}cx^2, \quad 0 \leq x \leq 1$$

subject to the boundary conditions

$$y(0) = y^{(1)}(0) = 1, \quad y(1) = 1.5 + \sinh(1), \quad y^{(1)}(1) = 1 + \cosh(1),$$

via orthonormalization in 1975 [12]. Zhu [28] introduced optimal quartic spline collocation methods for the numerical solution of this problem based on perturbation technique which gives rise to two optimal quartic spline one step and three step collocation methods. Van Daele et al. [26] constructed a new second order method for solving the boundary value problems (1.1) with the boundary conditions involving first derivatives based on nonpolynomial spline function. Siraj et al. [19,20] and the references therein solved a system of third order boundary value problem and fourth order two point boundary value problem using nonpolynomial spline functions. Ramadan et al. [8–10] and Zahra [27] established several methods based on nonpolynomial spline to generate second, fourth and sixth order convergent methods for the numerical solution of second, fourth, sixth order boundary value problems as well as high order boundary value problems. A family of methods based on exponentially and trigonometrically-fitted multistep of several orders are developed for the efficient solution of the Schrödinger equation, for details, we may refer to papers and the references therein [3,6,7, 15–18].

The aim of this paper is to construct a new spline method that is based on a exponential spline function of the form $ae^{kx} + be^{-kx} + p_{n-2}(x)$, with $p_{n-2}(x) = \sum_{i=0}^{n-2} m_i x^i$ is an ordinary polynomial of degree $n-2$ and an exponential part to develop numerical methods for obtaining smooth approximations for the solution of the problem (1.1) and (1.2). According to [5] the space $T_n = \text{span}\{1, x, x^2, x^3, \dots, x^n, e^{kx}, e^{-kx}\}$ generates an extended complete Chebyshev space on $[a, b]$. Thus it is possible to construct spline approximation that has a polynomial part and an exponential part. Spline function proposed in the present paper has the form $T_5 = \text{span}\{1, x, x^2, x^3, e^{kx}, e^{-kx}\}$ where k is free parameter which can be real or pure imaginary and which will be used to raise the accuracy of the method [5]. Thus in each subinterval $x_i \leq x \leq x_{i+1}$, we have:

$$T_5 = \text{span}\{1, x, x^2, x^3, e^{kx}, e^{-kx}\} \quad \text{or} \quad T_5 = \text{span}\{1, x, x^2, x^3, \cos kx, \sin kx\} \quad \text{or} \quad T_5 = \text{span}\{1, x, x^2, x^3, x^4, x^5\} \quad (\text{when } k = 0).$$

This approach has its own advantages in comparison with finite difference methods [21]. For example, once the solution has been computed, the information needed for spline interpolation between mesh points is available. This is important when the solution of the boundary value problem is required at different locations in interval $[a, b]$. This approach has added advantage that it not only provides continuous approximations to $y(x)$, but also for $y^{(1)}(x)$, $y^{(2)}(x)$ and higher derivatives at every point of the range of integration. Also, C^∞ -differentiability of the exponential part of the mixed splines compensates for loss of smoothness inherited by polynomial spline.

The paper is organized as follows: In sections 2, we introduce the exponential spline and derive our method. The method is formulated in a matrix form in section 3. Convergence analysis for second, fourth and sixth orders methods is established in section 4. Numerical results are presented to illustrate the applicability and accuracy in section 5. Finally, section 6 is devoted to conclude the results of the proposed methods.

2. Consistency relations

In order to develop the nonpolynomial spline approximation for the differential Eq. (1.1) along with the boundary condition (1.2), the interval $[a, b]$ is divided into n equal parts, using the grid points $x_i = a + ih$, $i = 0, 1, \dots, n$, $x_0 = a$, $x_n = b$ and $h = \frac{b-a}{n}$. The method for the solution of the boundary value problem (1.1) and (1.2) is developed using the exponential spline:

$$E_i(x) = a_i e^{\tau(x-x_i)} + b_i e^{-\tau(x-x_i)} + c_i(x-x_i)^3 + d_i(x-x_i)^2 + e_i(x-x_i) + f_i, \quad i = 0, 1, \dots, n, \quad (2.1)$$

where a_i , b_i , c_i , d_i , e_i and f_i are constants and k is a parameter which will be used to raise the accuracy of the method. Thus, our mixed spline is now defined by the relations:

$$\begin{aligned} \text{(i)} \quad & S(x) = E_i(x), x \in [x_i, x_{i+1}], \quad i = 0, 1, \dots, n \quad \text{and} \\ \text{(ii)} \quad & S(x) \in C^4[a, b]. \end{aligned} \quad (2.2)$$

Let S_i be an approximation to $u_i = u(x_i)$ obtained by the segment $E_i(x)$ of the mixed spline function passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}) given by Eq. (2.1). To obtain necessary conditions for the coefficients introduced in Eq. (2.1), it is not only required that $E_i(x)$ satisfies Eq. (1.1) at x_i and x_{i+1} and that the boundary conditions Eq. (1.2) are fulfilled, but also

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