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A nonlinear viscoelastic fractional derivative model of infant hydrocephalus

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ABSTRACT

Infant communicating hydrocephalus is a clinical condition where the cerebral ventricles become enlarged causing the developing brain parenchyma of the newborn to be displaced outwards into the soft, unfused skull. In this paper, a hyperelastic, fractional derivative viscoelastic model is derived to describe infant brain tissue under conditions consistent with the development of hydrocephalus. An incremental numerical technique is developed to determine the relationship between tissue deformation and applied pressure gradients. Using parameter values appropriate for infant parenchyma, it is shown that pressure gradients of the order of 1 mm Hg are sufficient to cause hydrocephalus. Predicting brain tissue deformations resulting from pressure gradients is of interest and relevance to the treatment and management of hydrocephalus, and to the best of our knowledge, this is the first time that results of this nature have been established.

1. Introduction

Hydrocephalus is a clinical condition of the brain caused by an imbalance in the absorption and production of cerebrospinal fluid (CSF) and characterized by enlargement of the central ventricular cavities. Communicating hydrocephalus is a puzzling case of hydrocephalus where there is no obstruction to the normal circulation of CSF, and hence no observable pressure gradient between the central ventricles and the subarachnoid space. Nevertheless, the cerebral ventricles still become enlarged and compress the brain parenchyma. Although hydrocephalus can occur at any age, until the last decade the most common occurrence of this clinical condition was in the pediatric population. However, with improving health care and an aging population, the situation has changed to a great degree and the occurrence of hydrocephalus has seen a marked increase amongst the elderly, whilst there has been a significant decline in the pediatric population.

A common assumption in mathematical models of hydrocephalus is that there exists a large pressure gradient between the cerebral ventricles and the subarachnoid space. This transmantle pressure gradient provides the necessary mechanical force required to compress the brain tissue, as evident in hydrocephalic brains. In infants, where the cranial sutures are unfused, such outward deformation of the brain parenchyma can cause the skull to expand if the condition is left untreated. In adults, where the cranial sutures are fused and the skull is rigid, this causes the parenchyma to be compressed against the skull.

The presence of such a large pressure gradient, however, has been questioned by Linninger et al. [10], Linninger and Penn [15]. They inserted pressure sensors into the ventricle, the parenchyma, and the subarachnoid space, and continuously recorded measurements in dogs with kaolin-induced hydrocephalus. Their measurements show no pressure differences

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between the ventricles and the parenchyma or the ventricles and the subarachnoid space. However, the sensitivity of their sensors was approximately 1 mm Hg, so that pressure differences below this threshold would not have been observed.

In this paper, by focusing on the case of infant communicating hydrocephalus, we will show that even pressure gradients on the order of 1 mm Hg are large enough to cause ventricular expansion, especially when the mechanical properties of the brain are degraded. Up to the age of two years, the infant brain undergoes rapid growth and development in a period known as the "brain growth spurt", and significant changes in the mechanical properties of the brain tissue occur as a result of this growth and development [18]. Recent work by Wilkie et al. [20] has shown that both the shear modulus and the steady-state elastic modulus of infant cerebrum are reduced when compared to the corresponding moduli of the adult cerebrum.

Both viscoelastic [11,3,16] and poroelastic [7,13,17] theories have been applied to the study of brain biomechanics. We use a fractional viscoelastic model due to their recent success in capturing the complex behaviour of brain tissue with a reasonably modest number of model parameters [2]. In our model, the brain tissue is represented by a Kelvin–Voigt viscoelastic solid, where a hyperelastic spring is coupled in parallel to a fractional viscoelastic dashpot. The nonlinear component allows for more accurate predictions of finite deformations compared to linear elastic models and the fractional derivatives incorporate the material deformation history into the stress–strain relation.

The outline of this paper is as follows: In Section 2 we derive the nonlinear viscoelastic fractional derivative model as well as analytic solutions under the assumption of small strains. In Section 3 numerical simulations for these analytic solutions are presented, for an iterative numerical technique which approximates the finite deformations observed in hydrocephalus by exploiting the incremental law of soft tissues [6, p. 238–239], and for the nonlinear model with first order derivatives. This mathematical analysis allows us to estimate the amount of ventricular expansion caused by a pressure gradient of 1 mm Hg. The paper concludes with a general discussion of the results in Section 4.

2. Mathematical analysis

In this section we derive the mathematical model for the nonlinear viscoelastic fractional derivative Kelvin–Voigt material and present analytic solutions to the resulting equation of motion for the case of small strains. The fully nonlinear equation of motion is presented at the end of the section. More details on the model derivation and solutions are presented in [21].

2.1. Model derivation

A simplified geometry of the hydrocephalic brain is used which is more amenable to analytical solutions. In fully developed hydrocephalus, when the cerebral ventricles are expanded, the brain is more akin to a cylindrical than spherical shape and so we set up our model as follows: consider a thick walled cylinder with inner radius R_1 and outer radius R_2 made of an incompressible fractional Kelvin–Voigt material, see Fig. 1(a).

At $t = 0$, the cylinder is in an undeformed state with a Lagrangian cylindrical co-ordinate system, (R,Θ,Z), and the unit direction vectors, $(\hat{e}_R, \hat{e}_\Theta, \hat{e}_Z)$. The internal surface is subjected to a pressure, $p_0(t)$, and the external surface is traction free, representing the unfused infant skull. There are no body forces. For $t > 0$, the cylinder is in a deformed state due to the applied boundary conditions and the Eulerian cylindrical co-ordinate system (r,θ,z) deforms with the material. The unit direction vectors of this space are $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$. Under the action of the internal pressure, $p_0(t)$, with the ends of the cylinder tethered, radially symmetric, planar deformations occur:

$$r = f(t,R), \quad \theta = \Theta, \quad z = Z, \tag{1}$$

where $f(t,R)$ is the deformation function to be determined. The radius vector of the deformed cylinder is thus $\vec{r}(t,R) = f(t,R)\hat{e}_r + z\hat{e}_z$.

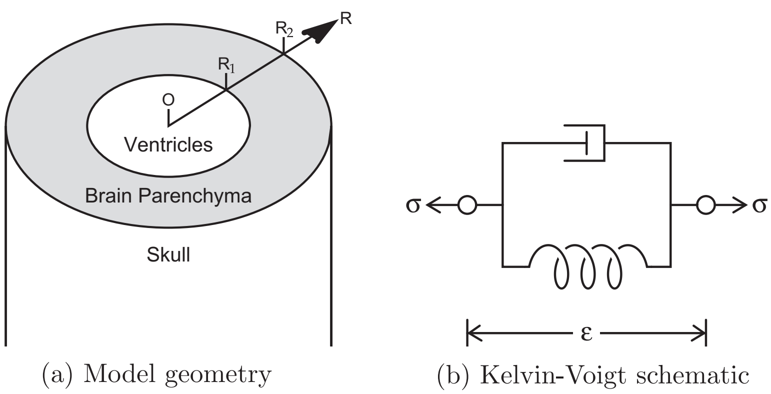

Fig. 1. The thick walled cylinder model geometry (a) and the Kelvin–Voigt viscoelastic material schematic (b).