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The explicit solution of the profit maximization problem with box-constrained inputs

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ABSTRACT

In this paper we study the profit-maximization problem, considering maximum constraints for the general case of m-inputs and using the Cobb–Douglas model for the production function. To do so, we previously study the firm's cost minimization problem, proposing an equivalent infimal convolution problem for exponential-type functions. This study provides an analytical expression of the production cost function, which is found to be a piece-wise potential. Moreover, we prove that this solution belongs to class C^1. Using this cost function, we obtain the explicit expression of maximum profit. Finally, we illustrate the results obtained in this paper with an example.

1. Introduction

One of the most important issues for firms in the field of microeconomics [1] is the profit-maximization problem. In this paper we consider a firm that operates under perfect competition, i.e. its prices are independent of the firm's input and output decisions. Consider a firm employing a vector of inputs $\mathbf{x} \in \mathbb{R}_+^m$ to produce an output $y \in \mathbb{R}_+$, where $\mathbb{R}_+^m$, $\mathbb{R}_+$ are non-negative m- and 1-dimensional Euclidean spaces, respectively. Let $P(\mathbf{x})$ be the feasible output set for the given input vector $\mathbf{x}$ and $L(y)$ the input requirement set for a given output y. Now, the technology set [2] is defined as

$$T = \left\{(\mathbf{x}, y) \in \mathbb{R}_+^{m+1}, \mathbf{x} \in L(y), y \in P(\mathbf{x})\right\}.$$

We assume that this set satisfies the following well-known regularity properties: closedness, non-emptiness, scarcity, and no free lunch.

Only on a few occasions have additional constraints been employed in the literature; for example, an expenditure constraint is considered in [3]. Most classical studies, however, simplify resource utilization without considering constraints on input usage. In this paper we establish, for the first time, a box-constrained profit-maximization problem, considering maximum constraints for the inputs.

Generally, the profit maximization problem can be formulated in the following way: the firm chooses inputs and output in order to maximize profits π (where profits are revenue minus costs), subject to technology constraints (i.e. the relationship between inputs and output):

$$\begin{aligned} &\pi(p, \mathbf{w}) = \max_{\mathbf{x},y} py - \mathbf{wx}), \\ \text{s.t.} \quad &y = f(\mathbf{x}), \\ &(\mathbf{x}, y) \in T, \\ &0 \leqslant x_i \leqslant M_i; \quad i = 1, \ldots, m, \end{aligned} \tag{1}$$

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where p is the price of the output, $\mathbf{w} \in \mathbb{R}^m$ are the vector prices of the inputs, M_i the maximum constraints for the inputs, and $f(\mathbf{x})$ is the production function, which is a continuous, strictly increasing and strictly quasiconcave function. In this paper we consider an m-input Cobb–Douglas production function [4], [5]:

$$y = f(\mathbf{x}) = A\prod_{i=1}^{m} x_i^{\alpha_i}.$$

There are two ways of solving this problem: (i) we can either formulate the problem maximizing over the input quantities, with plain Lagrange/Kuhn–Tucker or substituting the constraint in the objective function; or (ii) we formulate the problem using a minimum cost function and then maximize over the output quantity.

In this paper we use this short-cut-via-cost minimization. Note that if a firm is maximizing its profits and decides to offer a level of production y, it must be minimizing the cost of producing this output. Otherwise, a cheaper way of obtaining y production units would exist, which would mean that the firm is not be maximizing its profits. Namely: profit max implies cost min.

On the other hand, the profit-maximization problem has traditionally been solved by differentiating the variable x_i. Nevertheless, some authors avoid using total differentiation of first-order conditions, indicating that this gives rise to complicated equations which are difficult to handle. For example, [6] and [7] employ geometric programming to derive the maximal profit for the profit function. In the present paper we obtain the analytical and explicit formulas using the classical method of calculation.

The following are common problems than can arise: (i) The production function may not be differentiable, in which case we cannot take first-order conditions. (ii) The first-order conditions given above assume an interior solution, but we must also consider boundary solutions. (iii) A profit maximizing plan might not exist. (iv) The profit maximizing production plan might not be unique. In this paper we prove, under certain assumptions, the existence and uniqueness of the solution and that it belongs to class C^1.

The paper is organized as follows. Our box-constrained profit-maximization problem is solved in two stages: we first determine how to minimize the costs of producing each amount y and then what amount of production actually maximizes profits.

In the next section we present the box-constrained cost-minimization problem. By changing certain variables, we then transform it into a non-linear (exponential) separable programming problem [8], which we state as a constrained infimal convolution problem [9]. In Section 3, we provide a number of basic definitions and develop all the mathematical results necessary for the solution of the infimal convolution problem. Section 4 presents the optimal solution of the box-constrained cost-minimization problem. In Section 5, we obtain the optimal solution of the box-constrained profit-maximization problem to then discuss the results of a numerical example in Section 6. Finally, Section 7 summarizes the main conclusions of our research.

2. Cost-minimization problem

In this section we first present the classic firm's cost-minimization problem. This problem can be expressed as follows: produce a given output y, and choose inputs to minimize its cost:

$$\begin{aligned} & c(\mathbf{w}, y) = \min_{\mathbf{x} \geqslant \mathbf{0}} \mathbf{w}\mathbf{x}, \\ \text{s.t.}\quad & f(\mathbf{x}) = y, \end{aligned} \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^m$ are the inputs and $\mathbf{w} \in \mathbb{R}^m$ are the factor prices. There are a number of different ways to mathematically express how inputs are transformed into output. In this paper we consider the general Cobb–Douglas production function

$$y = f(x) = A\prod_{i=1}^{m} x_i^{\alpha_i}$$

and we shall usually measure units so that the total factor productivity $A = 1$. The sum of α_i determines the returns to scale.

The formulas for the corresponding cost function $c(\mathbf{w},y)$ are well known [10] when the production function follows the Cobb–Douglas model:

$$c(\mathbf{w}, y) = \alpha y^{\frac{1}{\alpha}} \prod_{i=1}^{m} \left(\frac{w_i}{\alpha_i}\right)^{\frac{\alpha_i}{\alpha}}, \quad \text{with } \alpha = \sum_{i=1}^{m} \alpha_i.$$

These formulas, which can be obtained simply using the Lagrange multipliers method, present the drawback that they are not applicable when upper limit constraints are considered for the different inputs.

In this paper we establish the analytical expression for the cost function $c(\mathbf{w},y)$ using the Cobb–Douglas model, considering maximum constraints for the inputs. Our cost-minimization problem will be: