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On a class of backward doubly stochastic differential equations ☆

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ABSTRACT

In this paper, a new class of backward doubly stochastic differential equations is studied. This type of equations has a more general form of the forward Itô integrals compared to the ones which have been studied until now. We conclude that unique solutions of these equations can be represented with the help of solutions of the corresponding backward doubly stochastic differential equations, considered earlier in paper [5] by Pardoux and Peng. Some comparison theorems are also given, as well as a probabilistic interpretation for solutions of the corresponding quasilinear stochastic partial differential equations.

1. Introduction

Nonlinear backward stochastic differential equations (henceforth BSDEs) with Brownian motions as noise source were introduced by Pardoux and Peng in their fundamental paper [1]. The motivation for the study of this kind of equations comes from the idea to provide probabilistic interpretation (generalized Feynman–Kac formula) for solutions of quasilinear partial differential equations (see [2]). Since then, BSDEs have been intensively developed with great interest and encountered in many fields of applied mathematics such as finance, stochastic games and optimal control (see El-Karoui et al. [3] and [4], for instance, and references therein).

In order to provide a probabilistic interpretation for the solutions of a special class of quasilinear partial differential equations, Pardoux and Peng [5] introduced in 1994 a new kind of BSDEs, called backward doubly stochastic differential equations (BDSDEs),

$$y_t = \xi + \int_t^T f(s, y_s, z_s)ds + \int_t^T g(s, y_s, z_s)dB_s - \int_t^T z_s dW_s, \quad t \in [0, T] \tag{1}$$

with two different directions of stochastic integrals driven by independent Brownian motions B_t and W_t. The integral with respect to $\{B_t\}$ is a "backward" Itô integral and the one with respect to $\{W_t\}$ is a standard forward Itô integral (both integrals are particular cases of the Itô–Skorohod integral (see Nualart and Pardoux [6]), while the solution is a pair (y_t, z_t) of processes adapted to the past of the Brownian motions.

Pardoux and Peng in [5] proved the existence-and-uniqueness theorem under uniform Lipschitz conditions on the coefficients f and g and for any square integrable terminal value ξ. Since then, many efforts have been made to relax conditions on f and g and to generalize in some sense Eq. (1) (see, for example, Lin [7], Aman [8], Aman [9], N'zi [10,11], Boufoussi et al. [12], Ren et al. [13], Hu et al. [14], and references therein). Because of significant applications in finance, a particular interest is directed to reflected BDSDEs (see El Karoui [3], Bahlali et al. [15], Shi [16], Ren [17] and references therein). Likewise, comparison theorems, as an important and effective technique in the study of BDSDEs, has been recently developed (see Lin [7], Shi et al. [18], El Otmani [19], Zhang [20]), as a useful tool, for instance, in studying viscosity solutions of stochastic partial differential equations.

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Note that all the above cited papers refer to BDSDEs with the forward integral $-\int_t^T z_s dW_s$. Moreover, to the best of the authors' knowledge, there are no results about BDSDEs containing forward integrals of a more general form. The topic of the present paper is to fill this gap by studying the following BDSDE dependent on a process h, that is, for $t \in [0,T]$,

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s,Z_s)dB_s - \int_t^T [h(s,Y_s) + Z_s]dW_s. \tag{2}$$

Our main aim is to prove that the solution of Eq. (2) can be represented with the help of the solution of Eq. (1). This fact makes it possible for us to study the existence and uniqueness and comparison problems for Eq. (2), by using the appropriate results referring to Eq. (1), as well as the relations between the solutions of Eq. (2) and the corresponding quasilinear system of SPDEs. In that way, proofs of all the assertions in the paper are significantly shortened and they do not present only a formal extension of the ones referring to Eq. (1).

The paper is organized as follows: In the remainder of this section, we introduce some notations and notions about BDSDE (2). In Section 2, we give our main results – the existence-and-uniqueness theorems under both Lipschitz and non-Lipschitz conditions. This part of the paper is essentially based on the fundamental paper [5] by Pardoux and Peng and on [8] by Aman. In Section 3, we state comparison theorems under both Lipschitz and non-Lipschitz conditions and prove them very simply by applying the results from Section 2. We close the paper by Section 4, where BDSDE (2) is related to a system of quasilinear stochastic partial differential equations.

First, we usually denote that $|\cdot|$ is the Euclidean norm in $\mathbb{R}^k$ and $\|A\| = \sqrt{tr(A^\top A)}$ the Frobenius trace-norm for a matrix A in $\mathbb{R}^{k\times d}$, where $A^\top$ is the transpose of A. We also generally assume that all random variables and processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and that $[0,T]$ is an arbitrarily large fixed time duration. We assume that $\{W_t, t\in[0,T]\}$ and $\{B_t, t\in[0,T]\}$ are two mutually independent standard Brownian motions with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively. Denote that $\mathcal{N}$ is the class of $\mathbf{P}$-null sets of $\mathcal{F}$. For every $t\in[0,T]$, let us define

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B,$$

where for any process $\eta_t, \mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s, r\in[s,t]\}\vee\mathcal{N}$ and $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$. Since $\{\mathcal{F}_t^W, t\in[0,T]\}$ and $\{\mathcal{F}_{t,T}^B, t\in[0,T]\}$ are increasing and decreasing filtrations, respectively, the collection $\{\mathcal{F}_t, t\in[0,T]\}$ is neither increasing nor decreasing and, therefore, it does not constitute a filtration.

As usual, for any $n\in\mathbb{N}$, let $\mathcal{M}^2([0,T];\mathbb{R}^n)$ be the set of (class of $d\mathbf{P}\times dt$ a.e. equal) $\mathbb{R}^n$-valued jointly measurable stochastic processes $\{\varphi_t, t\in[0,T]\}$ satisfying:

(i) $\|\varphi\|_{\mathcal{M}^2} = \mathbf{E}\int_0^T|\varphi_t|^2dt < \infty$;

(ii) φ_t is $\mathcal{F}_t$-measurable for a.e. $t\in[0,T]$.

Similarly, let $\mathcal{S}^2([0,T];\mathbb{R}^n)$ be the set of continuous $\mathbb{R}^n$-valued stochastic processes satisfying:

(i) $\|\varphi\|_{\mathcal{S}^2} = \mathbf{E}\sup_{t\in[0,T]}|\varphi_t|^2 < \infty$;

(ii) φ_t is $\mathcal{F}_t$-measurable for any $t\in[0,T]$.

Throughout the paper, the following basic assumptions hold:

(H_0) The terminal value is a random variable $\xi\in L^2(\Omega,\mathcal{F}_T,\mathbf{P};\mathbb{R}^k)$ and random functions $f:\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^k, g:\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^{k\times l}, h:\Omega\times[0,T]\times\mathbb{R}^k\to\mathbb{R}^{k\times d}$ are jointly measurable and such that for any $(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$,

$$f(\cdot,y,z)\in\mathcal{M}^2([0,T];\mathbb{R}^k),$$
$$g(\cdot,y,z)\in\mathcal{M}^2([0,T];\mathbb{R}^{k\times l}),$$
$$h(\cdot,y)\in\mathcal{M}^2([0,T];\mathbb{R}^{k\times d}).$$

Definition 1. A solution of Eq. (2) is a pair of $\mathbb{R}^k\times\mathbb{R}^{k\times d}$-valued processes $(Y,Z)=\{(Y_t,Z_t),t\in[0,T]\}\in\mathcal{S}^2([0,T];\mathbb{R}^k)\times\mathcal{M}^2([0,T];\mathbb{R}^{k\times d})$ which satisfies Eq. (2).

Definition 2. A solution $\{(Y_t,Z_t), t\in[0,T]\}$ of Eq. (2) is said to be unique, if for any other solution $\{(\bar{Y}_t,\bar{Z}_t),t\in[0,T]\}$ it follows that $\mathbf{P}\{Y_t=\bar{Y}_t, t\in[0,T]\}=1$ and $\mathbf{E}\int_0^T\|Z_t-\bar{Z}_t\|^2dt=0$.

We need the so-called extension of Itô's formula in our investigation.

Lemma 1 (Pardoux, Peng [5]). *Let $\alpha\in\mathcal{S}^2([0,T];\mathbb{R}^k),\beta\in\mathcal{M}^2([0,T];\mathbb{R}^k),\gamma\in\mathcal{M}^2([0,T];\mathbb{R}^{k\times l}),\delta\in\mathcal{M}^2([0,T];\mathbb{R}^{k\times d})$ be such that*

$$\alpha_t = \alpha_0 + \int_0^t\beta_s ds + \int_0^t\gamma_s dB_s + \int_0^t\delta_s dW_s, \quad t\in[0,T].$$