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On the convergence of Schröder iteration functions for *p*th roots of complex numbers

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ABSTRACT

In this work a condition on the starting values that guarantees the convergence of the Schröder iteration functions of any order to a *p*th root of a complex number is given. Convergence results are derived from the properties of the Taylor series coefficients of a certain function. The theory is illustrated by some computer generated plots of the basins of attraction.

1. Introduction

Throughout the paper, we will assume p and j to be two integers greater or equal than 2 and w to be a given complex number not belonging to the closed negative real axis. The pth roots of w are the p solutions of the polynomial equation

$$z^p - w = 0. \tag{1.1}$$

Let $\theta = \arg(w) \in]-\pi, \pi[$ denote the argument of w. It is well-known that for $n = 0, 1, \ldots, p-1$ each wedge of the complex plane defined by

$$\mathcal{W}_n = \left\{ z \in \mathbb{C} : \frac{(2n-1)\pi + \theta}{p} < \arg(z) < \frac{(2n+1)\pi + \theta}{p} \right\}, \tag{1.2}$$

contains exactly one pth root of w.

Our interest in studying iterative methods for pth roots comes from the problem of computing matrix pth roots. This is currently an important focus for research [1,7–12] mainly because of its applications in control and finance. Since the eigenvalues of a matrix are complex (even when the matrix has only real entries), the iteration functions for pth roots of complex scalars can be extended to the matrix case.

Consider the complex function f defined by $f(z) = (1-z)^{1/p}$ and let $T_j(z)$ denote the Taylor polynomial of degree j of $f(z)$ at zero. For each $j = 2, 3, \ldots$, the pth roots of w are fixed points of

$$N_j(z) := zT_{j-1}(1 - wz^{-p}), \tag{1.3}$$

which is an iteration function with order of convergence at least j (see [4]). This means that there is an initial guess z_0 such that the sequence defined by

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$$z_{k+1} = N_j(z_k), \tag{1.4}$$

converges to a pth root of w with order of convergence at least j.

It was shown in [4, Lemma 3.1] that N_j coincide with the Schröder iteration functions associated to the polynomial equation (1.1), which compels us to refer to N_j as the *Schröder iteration functions* for the pth roots of w. We refer the reader to [14,15] for more details about Schröder iteration functions.

The Taylor polynomials T_j in (1.3) are given by

$$T_j(z) := \sum_{n=0}^{j} \left(-\frac{1}{p}\right)_n \frac{z^n}{n!}, \tag{1.5}$$

where $(a)_k := a(a+1)\cdots(a+k-1)$ and $(a)_0 = 1$ represent the rising factorial of the complex number a (Pochhammer symbol). Recall that the particular case $j = 2$ is nothing more than the Newton's method for finding the zeros of the function $z^p - w$:

$$N_2(z) = z\left(1 - \frac{1}{p}(1 - wz^{-p})\right).$$

We note that the function $f(z) = (1 - z)^{1/p}$ has a formal (binomial) series representation [5, p.37]:

$$f(z) = \sum_{n \geqslant 0} \left(-\frac{1}{p}\right)_n \frac{z^n}{n!}, \tag{1.6}$$

and it is absolutely convergent inside the unit circle.

A complex function that is involved in the expression of N_j is the so-called residual function

$$R(z) := 1 - wz^{-p}. \tag{1.7}$$

The successive terms of the sequence (1.4) can be related by means of the residual function (1.7):

$$R(z_{k+1}) = 1 - (1 - z_k)(T_j(z_k))^{-p}, \tag{1.8}$$

(see [4, Sec. 3]). Let us denote the function that corresponds the right hand side of (1.8) by

$$\widetilde{R}_j(z) = 1 - (1 - z)\big(T_j(z)\big)^{-p}. \tag{1.9}$$

This function will play an important role in our work.

In Section 2 we will prove our main result which is Theorem 2.1. It states that $\widetilde{R}_j(z)$ admits a representation by a power series at $z = 0$ that is convergent for any complex number z inside the unit circle and whose first j coefficients are null while the remaining ones are positive. In order to accomplish this we will need to ensure the analyticity of $\widetilde{R}_j(z)$ inside the unit circle as well as to recall some other known results. The aforementioned theorem will enable us to derive in Section 3 some convergence results on Schröder iteration functions for pth roots. In particular, we show that if the initial guess z_0 satisfies the condition $|R(z_0)| < 1$, then for any j the sequence (1.4) converges to a pth root of w with order of convergence j. We recall that the case $j = 2$ has already been proved by Guo [8] and the case $j = 3$ by the present authors [4]. Our theoretical results will be illustrated by some examples of basins of attraction generated in Matlab.

2. Series representation of $\widetilde{R}_j(z)$

Lemma 2.1. *The roots of the Taylor polynomial $T_j(z)$ given by (1.5) lie outside the unit circle and consequently $\widetilde{R}_j(z)$ is analytic for any z such that $|z| < 1$.*

Proof. For any complex number z such that $|z| < 1$, we successively have

$$|T_j(z)| = \left|1 + \sum_{v=1}^{j} \frac{(-1/p)_v}{v!} z^v\right| = \left|1 - \frac{1}{p}\sum_{v=1}^{j} \frac{(1-1/p)_{v-1}}{v!}\ z^v\right| \geqslant 1 - \frac{1}{p}\sum_{v=1}^{j} \frac{(1-1/p)_{v-1}}{v!}|z|^v > 1 - \frac{1}{p}\sum_{v=1}^{j} \frac{(1-1/p)_{v-1}}{v!}$$
$$\geqslant 1 - 1 + \frac{(1-1/p)_j}{j!},$$

whence $|T_j(z)| > 0$ which implies $T_j(z) \neq 0$. Inasmuch as $\widetilde{R}_j(z)$ is a rational function whose poles lie outside the unit circle, its analyticity inside this domain is guaranteed. □

Based on the *Faà di Bruno's formula* [5,6,13] it is possible to derive the expression (although a tricky one) of the nth derivative of the function $(T_j(z))^{-p}$ by means of the (partial) *Bell polynomials* [3], or, more precisely, through the so called *potential polynomials*. We recall the result:

Proposition 2.1 [5, p.141]. *Consider the function $G(z) = 1 + \sum_{n\geqslant 1} g_n \frac{z^n}{n!}$ where $g_n = \frac{d^n}{dz^n}G(z)|_{z=a}$, $n \geqslant 1$. For any integer number r, the nth order derivative of the power function $(G(z))^r$ at the point $z = a$ can be computed through the potential polynomials $P_n^{(r)}$:*