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Limit cycles and singular point quantities for a 3D Lotka–Volterra system

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ABSTRACT

Four limit cycles are constructed for a three dimensional Lotka–Volterra system. This gives a good example to the cyclicity of 3D Lotka–Volterra systems. A recursion formula for computation of the singular point quantities is given for the corresponding Hopf bifurcation equation. What is worth mentioning is that the expressions of focal values are simpler, and the formula is readily done with using computer symbol operation system such as Mathematica due to its linearity.

1. Introduction

We consider the existence of limit cycles for the three-dimensional (3D) Lotka–Volterra system:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i\left(r_i - \sum_{j=1}^{3} a_{ij}x_i\right), \quad i = 1,2,3. \tag{LV}$$

At times, the problem is restricted to the assumption of competitive conditions that $r_i > 0$, $a_{ij} > 0$ $(i,j = 1,2,3)$, which describe the relations of three species that share and compete for the same resources, habitat or territory (interference competition). And based on the remarkable result of Hirsch, Zeeman [1] defined a combinatorial equivalence relation on the set of all 3D competitive LV systems and identified 33 stable equivalence classes. Of these, only classes 26–31 can have limit cycles. In [2], Hofbauer and So conjectured the number of limit cycles is at most two for Eq. (LV) with the competitive conditions. The interesting conjecture has triggered a lot of research, more details can be seen in [3–6]. Xiao and Li [7] proved that the number is finite for of the 3D competitive LV system without a heteroclinic polycycle. At present the best result is the example of four limit cycles for class 27, given by Gyllenberg et al. [5]. However, the question of how many limit cycles can appear in Zeeman's six classes 26–31 remains open up till now.

As for the 3D LV generic or non-competitive system, the subject on limit cycles has been paid little attention to. In this paper, we consider a 3D non-competitive system as follows:

$$\dot{x} = \mathrm{diag}(x)A(x - E), \tag{1}$$

where $x = (x_1,x_2,x_3)$, $E = (1,1,1)$, and all a_{ij} are arbitrary real numbers. Obviously E is the unique positive equilibrium of system (1). We prove that there exist four small amplitude limit cycles generated via generic Hopf bifurcation of E. Mainly, the good method from [8] is applied to compute the singular point quantities for the corresponding Hopf bifurcation equation, which are algebraic equivalent to the corresponding focal values. The expressions of focal values are simpler, and due to

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its linearity, the algorithm is readily implemented by computer symbol operation system such as Mathematica. We also think that these are very helpful to solve the cyclicity of above 3D competitive LV systems completely.

2. An example with four limit cycles

Here we consider system (1) with the interaction matrix A as follows:

$$A = \begin{pmatrix} 0 & 0 & \lambda \\ h & -1 & \mu \\ 0 & n & 0 \end{pmatrix},$$

and transform the equilibrium E to the origin, set $\tilde{x} = x - E$, then system (1) takes the form

$$\dot{x} = \mathrm{diag}(E + x)Ax \tag{2}$$

we still use x_i instead of $\tilde{x}_i$ for $i = 1,2,3$. In order to obtain an example of system (1) with four limit cycles, we only need to investigate the existence of four limit cycles in system (2). Furthermore, we choose the interaction matrix A such that the origin of system (2) can generate a Hopf bifurcation. Then suppose A has a pair of purely imaginary eigenvalues $\pm\mathbf{i}\omega(\omega > 0)$ and one negative real eigenvalue. To satisfy the necessary eigenvalue conditions, we need

$$\mathrm{Det}(A) = (A_{11} + A_{11} + A_{33})\mathrm{tr}(A),$$

where $\mathrm{tr}(A) = \sum_{i=1}^{3} a_{ii},\ A_{11} = a_{22}a_{33} - a_{23}a_{32},\ A_{22} = a_{11}a_{33} - a_{13}a_{31},\ A_{11} = a_{22}a_{11} - a_{12}a_{21}$. It yields that $\mu = h\lambda$, i.e. $\lambda = -\omega^2/\omega^2 hn$. Thus one can construct a matrix P which transforms A to be a block-diagonal one, i.e. using the transformation $x = Py$, such that

$$P^{-1}AP = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{where } P = \begin{pmatrix} \omega/(hn) & 0 & \omega^2/(hn) \\ \omega/n & 0 & -1/n \\ 0 & 1 & 1 \end{pmatrix}.$$

And more, after a time scaling: $t \to t/\omega$, we can get a new system from the system (2):

$$\dot{y} = \frac{1}{\omega}[P^{-1} * \mathrm{diag}(Py + E)APy] = \frac{1}{\omega}[P^{-1}AP\ y + P^{-1} * \mathrm{diag}(P\ y)AP\ y], \tag{3}$$

where $y = (y_1, y_2, y_3)$.

In order to investigate Hopf bifurcation of system (3), we can apply the formal series method given in [8]. Firstly by means of transformation: $y_1 = (z + w)/2$, $y_2 = (z - w)\mathbf{i}/2$, $y_3 = u$, $t = -T\mathbf{i}$, the system (3) can also be transformed into the following complex system:

$$\begin{cases} \frac{dz}{d}T = z + a_{101}uz + a_{011}uw + a_{200}z^2 + a_{020}w^2 + a_{002}u^2 = Z \\ \frac{dw}{dT} = -(w + b_{011}uz + b_{101}uw + b_{020}z^2 + b_{200}w^2 + b_{002}u^2) = -W \\ \frac{du}{dT} = \frac{\mathbf{i}}{\omega}u + d_{101}uz + d_{011}uw + d_{200}z^2 + d_{020}w^2 + d_{002}u^2 = U \end{cases} \tag{4}$$

where $u \in \mathrm{R}$, $z, w, T \in \mathrm{C}$, and

$$a_{200} = \frac{(hn - \omega^2 + hn\omega^2) + \mathbf{i}(\omega + h\omega^3)}{2hn(1 + \omega^2)}, \quad a_{020} = \frac{-(\omega + h\omega^3) - \mathbf{i}(hn - \omega^2 + hn\omega^2)}{2hn(1 + \omega^2)}, \quad a_{002} = \frac{(\omega + h\omega^3) + \mathbf{i}(h + hn + hn\omega^2 - \omega^4)}{2hn\omega(1 + \omega^2)},$$

$$a_{011} = \frac{-h(1 + n)\omega + \mathbf{i}(hn - \omega^2)}{2hn\omega}, \quad a_{101} = \frac{\omega(h + hn + \omega^2(2 - h + hn)) + \mathbf{i}(hn + \omega^2(1 - 2h + hn - \omega^2))}{2hn\omega(1 + \omega^2)},$$

$$b_{kjl} = \bar{a}_{kjl} \quad (kjl = 200, 020, 002, 011, 101),$$

$$d_{200} = \frac{(h - 1)\omega^2}{hn(1 + \omega^2)}, \quad d_{002} = \frac{\mathbf{i}(\omega^4 - h)}{hn\omega(1 + \omega^2)}, \quad d_{011} = \frac{(h + \omega^2) + \mathbf{i}(h\omega + \omega^3)}{hn(1 + \omega^2)}, \quad d_{101} = \frac{-(h + \omega^2) + \mathbf{i}(h\omega + \omega^3)}{hn(1 + \omega^2)}, \quad d_{020} = -d_{200}.$$