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# On existence of homoclinic orbits for some types of autonomous quadratic systems of differential equations

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ABSTRACT

The new existence conditions of homoclinic orbits for the system of ordinary quadratic differential equations are founded. Further, the realization of these conditions together with the Shilnikov Homoclinic Theorem guarantees the existence of a chaotic attractor at 3D autonomous quadratic system. Examples of the chaotic attractors are given.



## 1. Introduction

For many decades a chaotic behavior of dynamic systems remains in the focus of mathematicians, physicists and engineers. There are hundreds publications, in which different forms of this phenomenon is considered [1–4]. However, there are only a few publications, in which (from the mathematical point of view) the existence of chaotic dynamics is rigorously proved.

For example, the mathematically rigorous proof of the chaos existence in a modified Lorenz systems is presented in papers [5,6,9]. The authors use the theory of Shilnikov bifurcations of homoclinic and heteroclinic orbits. In [8,10,11,17] the study of Lorenz-type attractors has continued. Although these objects are deeply investigated, many unresolved questions still remain. As it is indicated in [6], a numerical evidence may occasionally be misleading, since computer simulations have finite precision and experimental measurements have finite ranges in the time or frequency domain. The witnessed behavior may be an artifact of the observation device due to physical limitations. Thus, a rigorous proof is often necessary for full understanding of chaotic dynamics in various nonlinear dynamic systems. Even for the Lorenz attractor, first discovery of chaos which has been extensively studied for over 40 years, only recently rigorous proof was obtained.

To investigate the chaotic dynamic in a partial differential equation one has to establish the existence of a chaotic attractor. The main problem here is an infinite dimensionality of the state space [7]. For an infinite dimensional state space the application of classical tools of the bifurcation theory is impossible.

As it is shown in the recent publications [12–16], [18–20] the basic tools in establishing presence of chaotic attractor for 3D system of autonomous quadratic differential equations are the Shilnikov Theorems.

A main contribution to the solution of a classification problem of chaotic attractors has been made in [16]. It was indicated that a large class of chaotic systems can be divided into the following four types: chaos of the homoclinic orbit type; chaos of the heteroclinic orbit type; chaos of the hybrid type; i.e. those with both homoclinic and heteroclinic orbits; chaos of other types. The simplest possible forms of chaotic systems were found for each type of chaos. In particular, several novel chaotic attractors were found, e.g. a hybrid-type chaotic attractor with three equilibria, with heteroclinic orbit and one homoclinic orbit, and a 4-scroll chaotic attractor with five equilibria and two heteroclinic orbits.

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In the present work, a criterion for the existence homoclinic orbits, without restriction on number nonlinearities, is establish for a large class of quadratic systems of the differential equations. As a basic tools we also use the Shilnikov Theorems.

We denote by $\mathbb{R}^n$ a real space of dimension $n$. Let $\mathbf{x}^T = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be an unknown vector where coordinates are functions of time $t$. Let also $A = (a_{ij}), B_1, \ldots, B_n \in \mathbb{R}^{n\times n}$ be real matrices and let the matrices $B_1,\ldots,B_n$ be symmetrical.

Consider the system of ordinary quadratic differential equations

$$\begin{cases} \dot{x}_1(t) = \sum\limits_{j=1}^{n} a_{1j}x_j(t) + \mathbf{x}^T(t)B_1\mathbf{x}(t) \equiv f_1(\mathbf{x}(t)), \\ \cdots, \\ \dot{x}_n(t) = \sum\limits_{j=1}^{n} a_{nj}x_j(t) + \mathbf{x}^T(t)B_n\mathbf{x}(t) \equiv f_n(\mathbf{x}(t)) \end{cases} \tag{1}$$

of order $n$ with the vector of initial values $\mathbf{x}^T(0) = (x_{10},\ldots,x_{n0})$.

Let us introduce some notations and definitions. Let $Q \subset \mathbb{R}^n$ be a compact (bounded and closed) set containing the origin. Symbol $\mathbf{x}(t,\mathbf{x}_0)$ denotes the solution (the trajectory) of system (1) satisfying the initial condition $\mathbf{x}(0,\mathbf{x}_0) = \mathbf{x}_0$. Further, we denote the distance between any vector $\mathbf{x}_k$ and $Q$ by $d(\mathbf{x}_k, Q) = \inf_{\mathbf{x}\in Q}\|\mathbf{x}_k - \mathbf{x}\|$.

**Definition 1** [21]. If there exists a compact set $Q \subset \mathbb{R}^n$ such that

$$\forall \mathbf{x}_0 \in \mathbb{R}^n, \lim_{t\to\infty} d(\mathbf{x}(t,\mathbf{x}_0), Q) = 0,$$

then we call $Q$ a globally attractive set of system (1). If

$$\forall \mathbf{x}_0 \in P \subset \mathbb{R}^n, \mathbf{x}(t,\mathbf{x}_0) \subseteq P,$$

then $P$ is called positive invariant set of system (1).

**Definition 2** [22]. A bounded trajectory $\mathbf{x}(t,\mathbf{x}_0)$ of system (1) is called a homoclinic orbit if the trajectory converges to the same equilibrium point as $t \to \pm\infty$.

Let $\mathbf{x}_e \in \mathbb{R}^n$ be an equilibrium point of system (1). Denote by:

$$D(\mathbf{x}_e) = (\partial f_i(\mathbf{x})/\partial x_j)(\mathbf{x}_e) \in \mathbb{R}^{n\times n}$$

the Jacobian matrix of the function $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}),\ldots,f_n(\mathbf{x}))^T$ in the equilibrium point $\mathbf{x}_e$; $i, j = 1,\ldots,n$.

**Theorem 1** (Shilnikov Homoclinic Theorem [22]). *Let $n = 3$, and let $\alpha$, $\beta \pm i\gamma$ be the eigenvalues of the matrix $D(\mathbf{x}_e)$, where $\alpha, \beta, \gamma \in \mathbb{R}, \alpha\beta < 0$, and $\gamma \neq 0$ (the equilibrium point is a saddle focus).*

*Suppose that the following conditions are fulfilled:*

(1) $|\alpha| > |\beta|$;
(2) *there exists a homoclinic orbit connected at $\mathbf{x}_e$.*

*Then:*

(1) *in a neighborhood of the homoclinic orbit there is a countable number of Smale horseshoes in discrete dynamics of system (1);*
(2) *for any sufficiently small $C^1$-perturbation $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}),\ldots,g_n(\mathbf{x}))^T$ of the function $\mathbf{f}(\mathbf{x})$ in system (1) the perturbed system $\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}) \in \mathbb{R}^n$ has at least a finite number of Smale horseshoes in the discrete dynamic defined near the homoclinic orbit;*
(3) *both the original system (1) and the perturbed system $\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x})$ have the horseshoe type of chaos.*

## 2. A triangular form of the quadratic system differential equations

Consider the following homogeneous system of the quadratic differential equations:

$$\begin{cases} \dot{x}_1(t) = \mathbf{x}^T(t)B_1\mathbf{x}(t), \\ \cdots, \\ \dot{x}_n(t) = \mathbf{x}^T(t)B_n\mathbf{x}(t) \end{cases} \tag{2}$$

with the vector of initial values $\mathbf{x}^T(0)$.

Note that in the system (2) any quadratic form can be uniquely presented as the sum

$$\mathbf{x}^T B_{i+1}\mathbf{x} = U_{1,i+1}(x_1,\ldots,x_i) + U_{2,i+1}(x_1,\ldots,x_n),$$