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On the solution of the Abel equation of the second kind by the shifted Chebyshev polynomials

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ABSTRACT

This paper presents a new approximate method of Abel differential equation. By using the shifted Chebyshev expansion of the unknown function, Abel differential equation is approximately transformed to a system of nonlinear equations for the unknown coefficients. A desired solution can be determined by solving the resulting nonlinear system. This method gives a simple and closed form of approximate solution of Abel differential equation. The solution is calculated in the form of a series with easily computable components. The numerical results show the effectiveness of the method for this type of equation. Comparing the methodology with some known techniques shows that the present approach is relatively easy and highly accurate.

1. Introduction

In this paper we consider the nonlinear Abel differential equation of the second type [1]

$$[A_1(t)+A_2(t)y(t)]y'(t)+B(t)y(t)+C(t)y^2(t)+D(t)y^3(t)=E(t) \tag{1}$$

with the mixed condition

$$\alpha_1 y(a)+\beta_1 y(b)+\gamma_1 y(c)+\alpha_2 y^2(a)+\beta_2 y^2(b)+\gamma_2 y^2(c)\gamma_2 y^2(c)+\alpha_3 y^3(a)+\beta_3 y^3(c)=\lambda \tag{2}$$

and the solution is expressed in the form

$$y(t)=\sum_{n=0}^{N}{}' a_n T_n^*(t), \quad T_n^*(t)=\cos(n\theta), \quad 2t-1=\cos\theta, \quad 0\leqslant t\leqslant 1, \tag{3}$$

where $T_n^*(t)$ denotes the shifted Chebyshev polynomials of the first kind, $\sum'$ denotes a sum whose first term is halved, a_n $(0\leqslant n\leqslant N)$ are unknown Chebyshev coefficients and N is chosen any positive integer such that $N\geqslant m$.

The nonlinear differential equations are essential tools for modelling many physical situations: chemical reactions, spring-mass systems, bending of beams and so forth. These equations have also demonstrated their usefulness in ecology and economics. Thus, the solution methods for these equations are of great importance to engineers and scientists. Although many important differential equations can be solved by well known analytical techniques, a greater number of physically significant differential equations cannot be solved. So far, many approaches have been proposed for determining the numerical solution to Abel equation [2]. For example, Fettis [3] proposed a numerical form of the solution to Abel equation by using the Gauss–Jacobi quadrature rule. Piessens and Verbaeten [4] and Piessens [5] developed an approximate solution to Abel

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equation by means of the integral transformations. Furthermore, Garza et al. [6] and Hall et al. [7] used the wavelet method to solve Abel equation. However, very few references have been found in technical literature dealing with Abel differential equations [8–18]. In this study, we present a new, simple approach for solving the approximate solution to Abel differential equations. By expanding the unknown function to be determined as a shifted Chebyshev polynomial, we can convert Abel differential equation to a system of nonlinear equations for the unknown function. An error analysis of approximate Abel differential equation is given. Finally, several examples are given to show the effectiveness of the present method.

The rest of this paper is organized as follows. Analysis of shifted Chebyshev collocation method and fundamental relations are presented in Section 2. The new scheme are based on shifted Chebyshev collocation method. Section 3 is devoted to the solution of Eqs. (1) and (2). In Section 4, we report our numerical finding and demonstrate the accuracy of the proposed numerical scheme by considering numerical example. Section 5 concludes this article with a brief summary.

2. Analysis of shifted Chebyshev collocation method

Let us consider the Abel differential Eq. (1) and find the truncated shifted Chebyshev series expansions of each term in expression (1) and their matrix representations. We first consider the desired solution $y(t)$ and its derivatives have truncated shifted Chebyshev series expansion of the form respectively,

$$y(t) = {\sum_{n=0}^{N}}' a_n T_n^*(t), \tag{4}$$

$$y^{(1)}(t) = {\sum_{r=0}^{N}}' a_n^{(1)} T_n^*(t), \quad t \in [0,1]. \tag{5}$$

Then the function defined in relation (4) can be written in the matrix form

$$y(t) = \mathbf{T}(t)\mathbf{A} \tag{6}$$

where

$$\mathbf{T}(t) = \lfloor T_0^*(t) \quad T_1^*(t) \ \cdots \ T_N^*(t) \rfloor \quad \mathbf{A} = [a_0 \quad a_1 \ \cdots \ a_N]^T. \tag{7}$$

Similarly, the matrix representation of function (5) becomes

$$y^{(1)}(t) = \mathbf{T}(t)\mathbf{A}^{(1)}. \tag{8}$$

Using the relation between the Chebyshev coefficient matrix $\mathbf{A}$ of $y(t)$ and the Chebyshev coefficient matrix $\mathbf{A}^{(k)}$ of $y^{(k)}(t)$ [14], we find the relation

$$\mathbf{A}^{(1)} = 4\ \mathbf{MA}, \tag{9}$$

where for odd N,

$$\mathbf{M} = \begin{bmatrix} 0 & 1/2 & 0 & 3/2 & 0 & 5/2 & \cdots & \frac{N}{2} \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)x(N+1)} \tag{10}$$

and for even N,

$$\mathbf{M} = \begin{bmatrix} 0 & 1/2 & 0 & 3/2 & 0 & 5/2 & \cdots & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & N \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)x(N+1)} . \tag{11}$$

Thus expression (9) becomes

$$y^{(1)}(t) = \mathbf{T}(t)\mathbf{A}^{(1)} = 4\mathbf{T}(t)\mathbf{MA}. \tag{12}$$

And after substituting the Chebyshev collocation points defined by