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Reexamination of an information geometric construction of entropic indicators of complexity

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ABSTRACT

Information geometry and inductive inference methods can be used to model dynamical systems in terms of their probabilistic description on curved statistical manifolds.

In this article, we present a formal conceptual reexamination of the information geometric construction of entropic indicators of complexity for statistical models. Specifically, we present conceptual advances in the interpretation of the information geometric entropy (IGE), a statistical indicator of temporal complexity (chaoticity) defined on curved statistical manifolds underlying the probabilistic dynamics of physical systems.

1. Introduction

The mystery of the origin of life and the unfolding of its evolution is perhaps the most fascinating topic that motivates the description and, to a certain extent, the understanding of the extremely elusive concept of complexity [1–3]. From a more pragmatic point of view, its description and understanding is also motivated by the question of how complex is quantum motion. This issue is of primary importance in quantum information science, having deep connections to entanglement and decoherence. However, our knowledge of the relations between complexity, dynamical stability, and chaoticity in a fully quantum domain is not satisfactory [4,5]. The concept of complexity is very difficult to define and its origin is not fully understood [6–8]. It is mainly for these reasons that several quantitative measures of complexity have appeared in the scientific literature [1,2]. In classical physics, measures of complexity are understood in a better satisfactory manner. The Kolmogorov–Sinai metric entropy [9,10], the sum of all positive Lyapunov exponents [11], is a powerful indicator of unpredictability in classical systems. It measures the algorithmic complexity of classical trajectories [12–15]. Other known measures of complexity are the logical depth [16], the thermodynamic depth [17], the computational complexity [18] and stochastic complexity [19] to name a few. For instance the logical and thermodynamic depths consider complex (roughly speaking) whatever can be reached only through a difficult path. Each one of these complexity measures captures to some degree our intuitive ideas about the meaning of complexity. Some of them just apply to computational tasks and unfortunately, only very few of them may be generalized so that their applications can be extended to actual physical processes. Ideally, a good definition of complexity should be mathematically rigorous as well as intuitive so as to allow for the analysis of complexity-related problems in computation theory and statistical physics. For obvious reasons, a quantitative measure of complexity is genuinely useful if its range of applicability is not limited to a few unrealistic applications. For similar reason, in order to properly define measures of complexity, the reasons for defining such a measure should be clearly stated as well as what feature the measure is intended to capture.

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One of the major goals of physics is modeling and predicting natural phenomena by using relevant information about the system of interest. Taking this statement seriously, it is reasonable to expect that the laws of physics should reflect the methods for manipulating information. Indeed, the less controversial opposite point of view may be considered where the laws of physics are used to manipulate information. This is exactly the point of view adopted in quantum information science where information is manipulated using the laws of quantum mechanics [20]. An alternative viewpoint may be explored where laws of physics are nothing but rules of inference [21]. In this view the laws of physics are not laws of nature but merely reflect the rules we follow when processing the information that happens to be relevant to the physical problem under consideration.

Inference is the process of drawing conclusions from available information. When the information available is sufficient to make unequivocal, unique assessments of truth, we speak of making deductions: on the basis of this or that information we deduce that a certain proposition is true. In cases where we do not have statements that lead to unequivocal conclusions, we speak of using inductive reasoning and the system for this reasoning is probability theory [22]. The word "induction" refers to the process of using limited information about a few special cases to draw conclusions about more general situations. Following this alternative line of thought, we extended the applicability of information geometric techniques [23] and inductive inference methods [24–28] to computational problems of interest in classical and quantum physics, especially with regard to complexity characterization of dynamical systems in terms of their probabilistic description on curved statistical manifolds. Moreover, we seek to identify relevant measures of chaoticity of such an information geometrodynamical approach to chaos (IGAC) [29–36].

In this article, we present a formal and conceptual reexamination of the information geometric entropy (IGE) [35], a statistical indicator of temporal complexity (chaoticity) of dynamical systems in terms of their probabilistic description using information geometry and inductive inference.

We emphasize we do not present here any new application of the IGAC (for instance, one of our most recent applications appears in [36]), however (and, most importantly) we do report some relevant conceptual advances in the interpretation of the IGE as a useful measure of complexity for statistical models suitable for probabilistic descriptions of dynamical systems.

The layout of this article is as follows. In Section 2, we briefly review our information geometric approach to the description of complex systems by using information geometry and inductive inference. In Section 3, we focus on the key-steps leading to the construction of the IGE and on its conceptual interpretation. Finally, in Section 4 we present our final remarks.

2. Complexity on curved manifolds

IGAC [29–34] is a theoretical framework developed to study chaos in informational geodesic flows describing physical systems. The reformulation of dynamics in terms of a geodesic problem allows for the application of a wide range of well-known geometric techniques to the investigation of the solution space and properties of the equations of motion. All dynamical information is collected into a single geometric object (namely, the manifold on which geodesic flow is induced) in which all the available manifest symmetries of the system are retained. For instance, integrability of the system is connected with existence of Killing vectors and tensors on this manifold. The sensitive dependence of trajectories on initial conditions, which is a key ingredient of chaos, can be investigated by using the equation of geodesic deviation. IGAC is the information geometric analogue of conventional geometrodynamical approaches [37,38] where the classical configuration space Γ_E is replaced by a statistical manifold $\mathcal{M}_S$ with the additional possibility of considering chaotic dynamics arising from non conformally flat metrics (the Jacobi metric is always conformally flat). It is an information geometric extension of the Jacobi geometrodynamics (the geometrization of a Hamiltonian system by transforming it to a geodesic flow [39]). In the Riemannian [37] and Finslerian [38] (a Finsler metric is obtained from a Riemannian metric by relaxing the requirement that the metric be quadratic on each tangent space) geometrodynamical approach to chaos in classical Hamiltonian systems, an active field of research concerns the possibility of finding a rigorous relation among the sectional curvature, the Lyapunov exponents, and the Kolmogorov–Sinai dynamical entropy (i.e., the sum of positive Lyapunov exponents) [40].

An n-dimensional $\mathbb{C}^\infty$ differentiable manifold (or more simply, a manifold) is a set of points $\mathcal{M}$ admitting coordinate systems $\mathcal{C}_\mathcal{M}$ and satisfies the following two conditions: (1) each element $c \in \mathcal{C}_\mathcal{M}$ is a one-to-one mapping from $\mathcal{M}$ to some open subset of $\mathbb{R}^n$; (2) For all $c \in \mathcal{C}_\mathcal{M}$, given any one-to-one mapping $\xi$from $\mathcal{M}$ to $\mathbb{R}^n$, we have that $\xi \in \mathcal{C}_\mathcal{M} \Longleftrightarrow \xi \circ c^{-1}$ is a $\mathbb{C}^\infty$ diffeomorphism. In this article, the points of $\mathcal{M}$ are probability distributions. Furthermore, we consider Riemannian manifolds $(\mathcal{M}, g)$. The Riemannian metric g is not naturally determined by the structure of $\mathcal{M}$ as a manifold. In principle, it is possible to consider an infinite number of Riemannian metrics on $\mathcal{M}$. A fundamental assumption in the information geometric framework is the choice of the Fisher-Rao information metric as the metric that underlies the Riemannian geometry of probability distributions [23,41,42], namely

$$g_{\mu\nu}(\Theta) \stackrel{\text{def}}{=} \int dXp(X|\Theta)\partial_\mu \log p(X|\Theta)\partial_\nu \log p(X|\Theta) = -\left(\frac{\partial^2 \mathcal{S}(\Theta', \Theta)}{\partial \Theta'^\mu \partial \Theta'^\nu}\right)_{\Theta'=\Theta}, \tag{1}$$

with $\mu,\nu = 1,\ldots, n$ for an n-dimensional manifold; $\partial_\mu = \frac{\partial}{\partial \Theta^\mu}$ and $\mathcal{S}(\Theta', \Theta)$ represents the logarithmic relative entropy [43],

$$\mathcal{S}(\Theta', \Theta) = -\int dXp(X|\Theta') \log\left(\frac{p(X|\Theta')}{p(X|\Theta)}\right). \tag{2}$$