Applied Mathematics and Computation 217 (2010) 3127–3136

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On an integral-type operator between weighted-type spaces and Bloch-type spaces on the unit ball

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ARTICLE INFO

Keywords:
Integral-type operator
Weighted-type space
Bloch-type space
Boundedness
Compactness
Essential norm
Unit ball

ABSTRACT

The boundedness and compactness of an integral-type operator, which has been recently introduced by the first author, between weighted-type spaces and Bloch-type spaces on the unit ball are studied here.

1. Introduction

Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$, $\mathbb{D}$ the open unit disk in $\mathbb{C}$ and $H(\mathbb{B})$ the class of all holomorphic functions on the unit ball. Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be points in $\mathbb{C}^n$ and $\langle z, w\rangle = \sum_{k=1}^{n} z_k \bar{w}_k$. For $f \in H(\mathbb{B})$ with the Taylor expansion $f(z) = \sum_{|\beta|\geqslant 0} a_\beta z^\beta$, let

$$\Re f(z) = \sum_{|\beta|\geqslant 0} \beta |a_\beta| z^\beta,$$

be the radial derivative of f, where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a multi-index, $|\beta| = \beta_1 + \cdots + \beta_n$ and $z^\beta = z_1^{\beta_1}, \ldots, z_n^{\beta_n}$. If X is a Banach space, then by B_X we denote the closed unit ball in X. Let μ be strictly positive continuous functions (weights) on the unit ball $\mathbb{B}$. A weight μ is called radial if $\mu(z) = \mu(|z|)$ for every $z \in \mathbb{B}$. Every radial weight μ which is nonincreasing with respect to $|z|$ and such that $\lim_{|z|\to 1-0} \mu(z) = 0$ is called a *typical weight.*

The class of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{H_\mu^\infty} = \sup_{z\in\mathbb{B}} \mu(z)|f(z)| < \infty,$$

where μ is a weight, is called the weighted-type space and is denoted by $H_\mu^\infty(\mathbb{B}) = H_\mu^\infty$. It is a Banach space with the norm $\|\cdot\|_{H_\mu^\infty}$. The little weighted-type space $H_{\mu,0}^\infty(\mathbb{B}) = H_{\mu,0}^\infty$ consists of all $f \in H(\mathbb{B})$ such that $\lim_{|z|\to 1-0} \mu(z)|f(z)| = 0$.

For a weight μ the *associate weight* $\tilde{\mu}$ ([3]) is defined as follows

$$\tilde{\mu}(z) = \frac{1}{\sup\{|f(z)| \,|\, f \in B_{H_\mu^\infty}\}} = \frac{1}{\|\delta_z\|_{(H_\mu^\infty)^*}}, \quad z \in \mathbb{B},$$

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doi:10.1016/j.amc.2010.08.045

where δ_z is the point evaluation at z. The associate weights are also continuous, $0 \leqslant \mu \leqslant \tilde{\mu}$ and for each $w \in \mathbb{B}$ there is an $f_w \in B_{H^\infty_\mu}$ such that

$$|f_w(w)| = 1/\tilde{\mu}(w).$$

If μ is typical, then the unit ball $B_{H^\infty_\mu}$ is the closure of $B_{H^\infty_{\mu,0}}$ for the compact open topology. Hence

$$\tilde{\mu}(z) = \frac{1}{\sup\{|f(z)|\,|f \in B_{H^\infty_{\mu,0}}\}}$$

and consequently for each $w \in \mathbb{B}$ there is an $f_w \in B_{H^\infty_{\mu,0}}$ such that $|f_w(w)| = 1/\tilde{\mu}(w)$. Of some special interest are weights μ such that $\tilde{\mu} = \mu$ (see, e.g. [2]).

The Bloch-type space $\mathcal{B}_\mu(\mathbb{B}) = \mathcal{B}_\mu$ consists of all $f \in H(\mathbb{B})$ such that

$$b_\mu(f) = \sup_{z\in\mathbb{B}} \mu(z)|\Re f(z)| < \infty,$$

where μ is a weight. The little Bloch-type space $\mathcal{B}_{\mu,0}(\mathbb{B}) = \mathcal{B}_{\mu,0}$ consists of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1} \mu(z)|\Re f(z)| = 0.$$

Both spaces $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu,0}$ are Banach with the norm $\|f\|_{\mathcal{B}_\mu} = |f(0)| + b_\mu(f)$, and $\mathcal{B}_{\mu,0}$ is a closed subspace of $\mathcal{B}_\mu$. Various Bloch-type spaces can be found, e.g, in [11,18,28,34,36].

A positive continuous function μ on the interval $[0,1)$ is called normal ([13]) if there is $\delta \in [0,1)$ and a and b, $0 < a < b$ such that

$$\frac{\mu(r)}{(1-r)^a} \quad \text{is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\mu(r)}{(1-r)^a} = 0;$$
$$\frac{\mu(r)}{(1-r)^b} \quad \text{is increasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\mu(r)}{(1-r)^b} = \infty.$$

If we say that a function $\mu : \mathbb{B} \to [0,\infty)$ is normal we also assume that it is radial.

Let φ be a holomorphic self-map of $\mathbb{B}$. For $f \in H(\mathbb{B})$ the composition operator is defined by

$$C_\varphi f(z) = f(\varphi(z)), \quad z \in \mathbb{B},$$

see, for example, [11,12,18,25,32] and the references therein.

In [20] the first author of this paper has extended products of integral and composition operators on $H(\mathbb{D})$ previously introduced by him and S. Li (see [8,10], as well as closely related operators in [7]) in the unit ball settings as follows (see also [23,26,30]). Assume $g \in H(\mathbb{B}),\ g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$, then we define an operator on the unit ball as follows

$$P^g_\varphi(f)(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(\mathbb{B}),\ z \in \mathbb{B}. \tag{1}$$

In this paper we continue to study operator P^g_φ by investigating the boundedness and compactness of the operator between the weighted-type spaces and the Bloch-type spaces. Results in Section 3 are closely related to those on weighted composition operators ([12]). Since there are some differences we will give more or less detailed proofs of our results in this section. In Section 4 we characterize the boundedness and compactness of $P^g_\varphi : \mathcal{B}_\mu(\text{or } \mathcal{B}_{\mu,0}) \to H^\infty_\nu(\text{or } H^\infty_{\nu,0})$. For some results on related integral-type operators in $\mathbb{C}^n$, see, e.g., [1,4–6,9,14–17,19,21,22,24,27,29,31,33–35,37] and the references therein.

2. Auxiliary results

In this section we quote several lemmas which are used in the proofs of the main results. The following lemma was proved in [20].

Lemma 1. *Assume that φ is a holomorphic self-map of $\mathbb{B}$, $g \in H(\mathbb{B})$ and $g(0) = 0$. Then for every $f \in H(\mathbb{B})$ it holds*

$$\Re[P^g_\varphi(f)](z) = f(\varphi(z))g(z).$$

The next characterization of compactness is proved in a standard way (see, e.g., the proofs of the corresponding lemmas in [6,15,16]), hence we omit its proof.

Lemma 2. *Assume that φ is a holomorphic self-map of $\mathbb{B}$, $g \in H(\mathbb{B})$, $g(0) = 0$, and ν and μ are weights. Let X and Y be one of the following spaces $H^\infty_\nu,\ H^\infty_{\nu,0},\ \mathcal{B}_\mu,\ \mathcal{B}_{\mu,0}$. Then the operator $P^g_\varphi : X \to Y$ is compact if and only if for every bounded sequence $(f_k)_{k\in\mathbb{N}} \subset X$ converging to 0 uniformly on compacts we have*

$$\lim_{k\to\infty} \|P^g_\varphi f_k\|_Y = 0.$$

The following result is well-known (see, e.g. [1,4]).