Applied Mathematics and Computation 217 (2010) 3355–3367

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A hybrid method using wavelets for the numerical solution of boundary value problems on the interval

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ARTICLE INFO

Keywords:
B-spline functions
Second order boundary problems
Wavelets
Multiresolution analysis
Wavelet-Galerkin

ABSTRACT

In this work, various aspects of wavelet-based methods for second order boundary value problems under Galerkin framework are investigated. Based on the B-spline multiresolution analysis (MRA) on the line we propose a hybrid method on the interval which combines different treatments for interior and boundary splines. By using this procedure, the MRA structure was conserved and hierarchical representations of the solution at different scales were obtained without much computational effort. Numerical examples are given to verify the effectiveness of the proposed method and the comparison with other techniques is presented.

1. Introduction

Since the mid 90s and due to their desirable properties, researchers have paid attention to wavelet analysis in solving differential equations. Wavelets provide a robust and accurate alternative to traditional methods and their advantage is really appreciated when they are applied to problems having localized singular behavior. The solution is approximated by a wavelet and scaling expansion, with the convenience that multi-scale and localization properties can be exploited. The choice of wavelet basis is governed by several factors including the desired order of numerical accuracy and computational effort.

Compactly supported wavelets introduced by Daubechies [1] combine orthogonality with localization and scaling properties and they have been used for the numerical approximation to differential equations solutions. There are a number of papers in this direction including solving the Dirichlet boundary value problem [2] and the development of different wavelet-based finite elements methods in structural mechanics [3–5].

On the other hand, it is well known that polynomial splines have the best approximation properties among all known wavelets of a given order and they have also been used in combination with finite element methods. For instance, Han et al. [6] used splines as interpolating functions to approximate displacements in structural mechanics problems and Xiang et al. [7] developed finite elements to solve plane elastomechanics and plate problems using B-spline wavelets on the interval. Unlike most other wavelet bases – as Daubechies –, splines have an explicit formula, which greatly facilitates their manipulation. They also allow for a transition from Haar's piecewise constant representation (spline of degree zero) to higher order splines in a very simple manner.

Another good feature of wavelet methods is the possibility to apply adaptive techniques: the combination of multiscale bases with finite element methods leads to adaptive refinement strategies such as the multiscale lifting method or the dynamically adaptive algorithm developed by Chen et al. [8] and Bindal et al. [9], respectively.

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2010.08.068

Some authors applied *wavelet collocation methods*, by using different wavelets in solving PDEs. Bertoluzza [11], uses autocorrelation functions of Daubechies compactly supported wavelets while Radunovic [10] uses spline wavelets. Moreover, adaptive procedures with cubic splines were introduced by Cai and Kumar et al. [12,13], allowing an optimization of the number of basis functions used for the solution of the problem. It is important to notice that collocation methods are efficient, but they require certain regularity conditions. In some cases, Galerkin method using variational equations and appropriate elemental functions, is a good alternative, producing an efficient regularization action. In weak formulations for a given equation, the approximation functions can be relatively less continuous and easier to construct [14].

In this paper, B-cubic-spline functions are used. Their basic properties, such as regularity and compact support, are recalled and weak formulations of Galerkin method using spline wavelets as basis functions are analyzed. The approximate solution for a general variational problem on an interval is then searched in spline form.

How to construct a hierarchy of spline subspaces for a linear differential operator is presented: interior and boundary splines on an interval are defined and described briefly. Different strategies can be considered according to including or not boundary splines.

In particular, for one-dimensional linear second order boundary problems, different approximations are analyzed and compared and a method combining variational equations with a collocation scheme is developed. If only variational equations are considered, the rate of convergence is perturbed at the ends of the interval (as a consequence of increasing the scale). An alternative to overcome this drawback, is to consider strong equations near the ends of the interval, replacing a fixed small number of boundary equations by others specially designed. In this sense, the method we propose is a *hybrid method* because it uses both, variational and collocation equations.

This hybrid method has good features since it could be applied at an initial scale j_0, and then, improve this j_0-approximation through a tool designed specially to increase the scale. In this way, we obtain a useful hierarchical representation of the solution. Wavelets play a crucial role in this refinement process, taking into account that they suggest the combination of the hybrid method with powerful adaptive techniques. This second step will be treated in a future work where an adaptive strategy will be developed and added to the hybrid increasing the scale process.

Numerical examples are used to demonstrate the applicability of the proposed method, whose approximate solutions are computed in scaling-spline form. These test problems were chosen to be simple, with known solutions, including a convection–diffusion problem with a boundary layer and a singularly perturbed reaction–diffusion equation. The approximations were validated and the convergence of the proposed method solutions are found to compare favourably to other numerical solutions.

The outline of the paper is as follows: Galerkin method using scaling functions as basis functions for a general differential operator is introduced in Section 2. In Section 3 we give a review and a summary of the basic information concerning Multiresolution Analysis (MRA) necessary for the development of the present work. In Section 4 we exhibit explicitly B-splines properties in general and also in a MRA framework. In Section 5 we propose a hybrid wavelet-Galerkin method to approximate the solution of a second order boundary value problem, combining variational and collocation equations. We also describe briefly how the solution could be improved by using wavelets. Numerical examples for this problem are described and analyzed in Section 6. A comparison of the proposed method in contrast with other numerical methods is also shown. In Section 7, conclusions and future work are presented.

2. Galerkin method

Mathematical models of problems appearing in science and engineering often take the form

$$Lu = f, \tag{1}$$

where L is a linear differential operator in certain function Hilbert space V. Since Eq. (1) cannot, in general, be solved exactly, one has to rely on approximation methods. As a first step towards this goal, an approximation $\tilde{u}$ of u which lies in certain finite dimensional subspace of V must be obtained. For this purpose, assuming that the Eq. (1) has a unique solution one could seek an approximation of the solution u in $V_N = \text{span}\{\Phi_1, \Phi_2, \ldots, \Phi_N\} \subset V$.

Let $\langle \cdot, \cdot \rangle$ be the inner product of the Hilbert space V. Note that $a(u, v) = \langle Lu, v \rangle$ defines a bilinear form on $V \times V$, so that the variational or weak formulation corresponding to the problem Eq. (1), is to seek $u \in V$, such that

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in V, \tag{2}$$

It is well known that if $a(\cdot, \cdot)$ is continuous, V-elliptic and $l(v) = \langle f, v \rangle$ is a continuous linear form in V, there exists a unique solution $u \in V$ of the problem Eq. (2), (Lax–Milgram theorem [15]).

Under these conditions, it is reasonable to look for $\tilde{u} \in V_N$ such that

$$\langle L\tilde{u}, \Phi_n \rangle = \langle f, \Phi_n \rangle \quad n = 1, 2, \ldots, N. \tag{3}$$

If we denote,

$$\tilde{u} = \sum_{k=1}^{N} \alpha_k \Phi_k, \tag{4}$$