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Composition operators from the weighted Bergman space to the *n*th weighted-type space on the upper half-plane

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ABSTRACT

The boundedness of the composition operator from the weighted Bergman space to, recently introduced by this author, the *n*th weighted-type space on the upper half-plane $\Pi_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ are characterized here.

1. Introduction

Let Π_+ be the upper half-plane in the complex plane $\mathbb{C}, H(\Pi_+)$ the space of all analytic functions on Π_+ and $dA(z)$ be the area measure on Π_+. For $0 < p < \infty$ and $\alpha \geqslant 0$, the weighted Bergman space $\mathcal{A}^p_\alpha(\Pi_+)$ consists of all $f \in H(\Pi_+)$ such that

$$\|f\|^p_{\mathcal{A}^p_\alpha(\Pi_+)} = \int_{\Pi_+} |f(z)|^p y^\alpha \, dA(z) < \infty,$$

where $y = \operatorname{Im} z$. When $p \geqslant 1$ the weighted Bergman space with the norm $\|\cdot\|_{\mathcal{A}^p_\alpha(\Pi_+)}$ is a Banach space.

Let $\mu(z)$ be a positive continuous function on a domain $X \subset \mathbb{C}$ (*weight*) and $n \in \mathbb{N}_0$ be fixed. The *n*th weighted-type space on X, denoted by $\mathcal{W}^{(n)}_\mu(X)$ consists of all $f \in H(X)$ such that

$$b_{\mathcal{W}^{(n)}_\mu(X)}(f) := \sup_{z\in X} u(z)|f^{(n)}(z)| < \infty.$$

This space was recently introduced by this author in [17].

For $n = 0$ the space becomes the weighted-type space $H^\infty_\mu(X)$, for $n = 1$ the Bloch-type space $\mathcal{B}_\mu(X)$, and for $n = 2$ the Zygmund-type space $\mathcal{Z}_\mu(X)$.

The quantity $b_{\mathcal{W}^{(n)}_\mu(X)}(f)$ is a seminorm on the *n*th weighted-type space $\mathcal{W}^{(n)}_\mu(X)$ and a norm on $\mathcal{W}^{(n)}_\mu(X)/\mathbb{P}_{n-1}$, where $\mathbb{P}_{n-1}$ is the set of all polynomials whose degrees are less than or equal to $n-1$. A natural norm on the *n*th weighted-type space can be introduced as follows

$$\|f\|_{\mathcal{W}^{(n)}_\mu(X)} = \sum_{j=0}^{n-1} |f^{(j)}(a)| + b_{\mathcal{W}^{(n)}_\mu(X)}(f),$$

where a is an element in X. With this norm the *n*th weighted-type space becomes a Banach space.

For $X = \Pi_+$ we obtain the space $\mathcal{W}^{(n)}_\mu(\Pi_+)$ on which the following norm can be introduced

$$\|f\|_{\mathcal{W}^{(n)}_\mu(\Pi_+)} := \sum_{j=0}^{n-1} |f^{(j)}(i)| + \sup_{z\in\Pi_+} \mu(z)|f^{(n)}(z)|.$$

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For $X = \mathbb{D}$ we get the space $\mathcal{W}_\mu^{(n)}(\mathbb{D})$, and a norm on it is introduced as follows

$$\|f\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} := \sum_{j=0}^{n-1} |f^{(j)}(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f^{(n)}(z)|.$$

Some information on Zygmund-type spaces on the unit disc (mostly with the weight $\mu(z) = (1 - |z|^2)$) and some operators on them, can be found, e.g., in [1,6–8,29] for the case of the upper half-plane see [3,17,18], while some information in the setting of the unit ball can be found, e.g., in [9,21,28]. This considerable interest in Zygmund-type spaces, as well as a necessity for unification of weighted-type, Bloch-type and Zygmund-type spaces, motivated us to introduce the nth weighted-type space.

Assume φ is a holomorphic self-map of X. The composition operator induced by φ is defined on $H(X)$ by

$$(C_\varphi f)(z) = f(\varphi(z)), \quad z \in X.$$

A typical problem is to provide function theoretic characterizations when φ induce bounded or compact composition operators between two given spaces of holomorphic functions, see, for example [2,11,18,26] and the references therein.

Motivated by Sharma et al. [10], in [17] the present author proved the following result:

Theorem A. *Assume $p \geqslant 1$ and φ is an analytic self-map of Π_+. Then $C_\varphi : H^p(\Pi_+) \to \mathcal{Z}_\infty(\Pi_+)$ is bounded if and only if*

$$\sup_{z\in\Pi_+} \frac{\mathrm{Im}\, z}{(\mathrm{Im}\ \varphi(z))^{2+\frac{1}{p}}} |\varphi'(z)|^2 < \infty$$

and

$$\sup_{z\in\Pi_+} \frac{\mathrm{Im}\, z}{(\mathrm{Im}\ \varphi(z))^{1+\frac{1}{p}}} |\varphi''(z)| < \infty.$$

Moreover, if the operator $C_\varphi : H^p(\Pi_+) \to \mathcal{Z}_\infty(\Pi_+)/\mathbb{P}_1$ is bounded, then

$$\|C_\varphi\|_{H^p(\Pi_+)\to\mathcal{Z}_\infty(\Pi_+)/\mathbb{P}_1} \asymp \sup_{z\in\Pi_+} \frac{\mathrm{Im}\, z}{(\mathrm{Im}\ \varphi(z))^{2+\frac{1}{p}}} |\varphi'(z)|^2 + \sup_{z\in\Pi_+} \frac{\mathrm{Im}\, z}{(\mathrm{Im}\ \varphi(z))^{1+\frac{1}{p}}} |\varphi''(z)|.$$

Motivated by Theorem A, the author of [3] characterized the boundedness of composition operators from the weighted Bergman spaces to the weighted-type, Bloch-type and Zygmund-type spaces, with the weight $\mu(z) = \mathrm{Im} z$, on the upper half-plane. On the other hand, in [18] we characterized the composition operators from the weighted Bergman space to the nth weighted spaces on the unit disc. The case $n = 0$ was previously treated, for example, in [12,16]. The case $n = 1$ was treated, for example, in [22].

Motivated by Jiang [3] and Stević [18] here we characterize the boundedness of composition operators from the weighted Bergman space to the nth weighted-type space on the upper half-plane for $n \in \mathbb{N}$. This paper can be regarded as a continuation of our investigations of composition, weighted composition, and related operators on spaces of holomorphic functions, see, e.g., [6–9,11–25].

Let X and Y be topological vector spaces whose topologies are given by translation-invariant metrics d_X and d_Y, respectively, and $T : X \to Y$ be a linear operator. It is said that T is *metrically bounded* if there exists a positive constant K such that

$$d_Y(Tf, 0) \leqslant K d_X(f, 0)$$

for all $f \in X$. When X and Y are Banach spaces, the metrically boundedness coincides with the usual definition of bounded operators between Banach spaces.

If Y is a Banach space then the quantity $\|C_\varphi\|_{A^p_\alpha(\Pi_+)\to Y}$ is defined as follows

$$\|C_\varphi\|_{A^p_\alpha(\Pi_+)\to Y} := \sup_{\|f\|_{A^p_\alpha(\Pi_+)}\leqslant 1} \|C_\varphi f\|_Y.$$

It is easy to see that this quantity is finite if and only if the operator $C_\varphi : A^p_\alpha(\Pi_+) \to Y$ is metrically bounded. For the case $p \geqslant 1$ this is the standard definition of the norm of the operator $C_\varphi : A^p_\alpha(\Pi_+) \to Y$, between two Banach spaces. If we say that an operator is bounded it means that it is metrically bounded.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \leqslant Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Auxiliary results

In this section we prove several auxiliary results which we use in the proofs of the main results. The first two lemmas should be folklore but we will give some proofs for reader's benefit and for the completeness (for related results see [4,5]).