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A numerical method for the dynamics and stability of spiral waves

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ABSTRACT

In this paper we propose a new method of investigating the change of dynamics in reaction–diffusion equations, which is based on approximating the Euclidian norm of state variables along with the introduction of phase space. Our method is simple in implementation and can be applied to study the dynamics of multiple spirals. The method is extended to study the stability of spiral waves by developing an algorithm which is applied to circular and meandering motions.

1. Introduction

The most striking phenomena in two-dimensional excitable media are the existence of spiral waves, their dynamics and the nature of these waves. Numerous studies in the area of spiral waves can be found in the literature. These include the appearance of spiral waves [1–7], their dynamics and simulation of the reaction-diffusion equations [8–15]. It is now well understood that the dynamics of spiral waves change with system parameter changes. This phenomenon is investigated by tracing the location of the spiral's tip in the excitable medium. It is observed that spiral waves either perform the circular (periodic) motion or meander (quasi-periodic). Our interest is to develop an alternative approach to study the change of dynamics in reaction–diffusion equations, by measuring the Euclidian norm. The approach requires a few lines of programming code and it is a very efficient way of studying the change in dynamics.

Of fundamental importance is the stability of spiral waves (rigidly rotating and meandering waves). The stability of rotating waves has been the subject of experimental and theoretical studies [16–18]. Our further objective in this paper is to study the stability of the waves. We perturb them from their original states and study their dynamical behaviour, using the approximated norm and through a numerical procedure.

The stability of rotating spiral waves has been investigated in a number of papers. Cowie and Rybicki [16] studied the propagation of a detonation wave front in a differentially rotating disk. They derived a partial differential equation and showed that, subject to certain boundary conditions, stable quasi-stationary spiral waves exist which rigidly rotate with a fixed pattern speed. Later Balbus [17] studied the propagating wave on rotating disks and focused on the local inclination of the front, giving some additional insight into the stability of these wave fronts. Balbus [17] also focused on the analytical treatment of the partial differential equation. Barkley [18] converted the time dependent reaction–diffusion equations to the steady state problem by studying the waves in an appropriate co-rotating (travelling) frame and observing the rotating (travelling) waves as steady state solutions. Then he used Krylov's subspace method followed by Newton's method to solve an eigenvalue problem.

Recently Bordyugov and Engel [19] developed a new numerical method of computing spiral waves and studying their stability. They transformed the time dependent partial differential equations to a system of ordinary differential equations by using Fourier series and then applying the continuation package AUTO [20] to compute the solution branches to detect stable and unstable solutions.

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Our methodology requires direct integration of the equations by approximating the equations using finite differences and writing down the numerical scheme which gives rise to the approximate solution of the equations. Having obtained the solution, we then approximate the Euclidian norm $\|u\|_2 = \left\{\int_0^L \int_0^L |u(x,y,t)|^2 dxdy\right\}^{\frac{1}{2}}$, which is a function of time t, to determine the change in dynamics and to study the stability of solutions using the proposed algorithm. The function $u(x,y,t)$ will be defined in Section 2. The upper limit L is the size of square domain where the function u is defined.

2. The approximation of Euclidian norm and the investigation of dynamics

The two state reaction–diffusion system of equations that we consider is the FitzHugh–Nagumo type model (Barkley model):

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nabla^2 u + \frac{1}{\epsilon} f(u, v),\\ \frac{\partial v}{\partial t} &= g(u, v),\end{aligned} \tag{1}$$

where $f(u,v) = u(1-u)(u - \frac{v+b}{a})$, and $g(u,v) = u - v$ are nonlinear functions of the state variables $u = u(x,y,t)$, $v = v(x,y,t)$, and the Laplacian is defined as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

The excitable medium is a two-dimensional square domain of size L with Neumann boundary conditions.

Throughout this paper we assume $L = 50$ and use Euler explicit finite difference method to solve system (1) by discretising the domain to equally spaced mesh points $x_i = ih$ and $y_j = jh$ where $i, j = 0, \ldots, 120$ and the steps at which the domain is discretised defined by $h = 50/120$. We omit the process of writing the approximated difference scheme here and refer the interested reader to any text on numerical analysis or previously published articles in this area [18,9]. Here we emphasise the technique of approximating the Euclidian norm of the variables u and v which is being used to describe the spiral dynamics. First we discretise the domain and solve the equations numerically, thus obtaining the functions u and v at the mesh points (x_i, y_j), from which the two-norm of the functions u and v can be approximated as follows:

$$s_1(t) = \|u\|_2 = \left\{\int_0^{50}\int_0^{50} |u(x,y,t)|^2 dxdy\right\}^{\frac{1}{2}} \approx \left\{h^2 \sum_{i=0}^{120}\sum_{j=0}^{120} |u(x_i, y_j, t)|^2\right\}^{\frac{1}{2}}, \tag{2}$$

where t represents the time at which the norm is calculated. The norm of function v is represented by $s_2(t)$ which can be approximated using the expression (2) by replacing u with v. The computer coding, for evaluating $s_1(t)$ and $s_2(t)$, is very simple and enables the norm of solutions to be determined efficiently.

To study the change in dynamics, we fix the physical parameters $b = 0.001$, $\epsilon = \frac{1}{50}$ with the time step $\Delta t = 0.01$ and calculate $s_1(t)$, $s_2(t)$ when the values of parameter a vary. We start with $a = 0.27$ and measure the norm when time progresses. For this value the motion is periodic as shown in Fig. 1(b) and (c). For the above value of a the size of excited region (spiral) is small as can be seen in Fig. 1(a). Thus one full rotation of the spiral tip takes a longer time. With our approach one can easily monitor the numerical data and obtain the period, T, of the pattern. In the case of $a = 0.27$ we obtain $T = 14.52$ which can also be easily identified from the Fig. 1(c). As we increase the value of a, the spiral grows. A fully resolved spiral for $a = 0.33$, at $t = 150$, is shown in Fig. 1(d). It can be seen that the spiral core has moved in the direction towards the centre of the domain. For $a = 0.33$ the pattern is still periodic which can be seen in Fig. 1(e) and (f). However, the spiral's growth has led to cycles with shorter periods. The growth of the spiral can be verified by monitoring the maximum value of $s_1(t)$, i.e. $\|s_1(t)\|_\infty = \max_{0 \leqslant t \leqslant 150} |s_1(t)|$, for $0.27 \leqslant a \leqslant 0.9$, as shown in Fig. 4 (see also the scales in Fig. 1(e) and (f) and compare these with the scales in Fig. 1(b) and (c)). We also monitor the period of the wave in the range $a = 0.27$ to $a = 0.33$ as illustrated in Fig. 2. It can be seen that when parameter a varies from 0.27 to 0.33 then the period decreases from 14.52 to 4.9 having approximately 10 units drop, indicative of a faster rotation of the spiral wave.

As parameter a increases to, say, $a = 0.35$ the spiral grows further towards the centre of the domain and the wave becomes much faster, inducing the second frequency of the pattern namely the frequency of drift. Using our approach, this phenomenon is represented in Fig. 3. It is obvious that the trajectory in Fig. 3(b) is not closing after each rotation. The drift frequency of the pattern will vanish with the increase in parameter a due to the linear growth of spiral (see Fig. 4) and the faster rotation speed. We elaborate this by studying the spiral rotation for $a = 0.64$ and $a = 0.65$. For these parameter values the maximum norm and the period become

$$\begin{aligned}&a = 0.64, \quad \|s_1(t)\|_\infty = 25.38, \quad T \approx 3.12;\\ &a = 0.65, \quad \|s_1(t)\|_\infty = 25.73, \quad T = 2.54.\end{aligned}$$

At $a = 0.64$ the wave meanders, while at $a = 0.65$ it approaches a periodic motion. Thus it is possible to approximate the period of its rotation using the numerical data. From the above we see that when the parameter a increases from 0.64 to 0.65,