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Properties of non-simultaneous blow-up in heat equations coupled via different localized sources

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ABSTRACT

This paper deals with $u_t = \Delta u + u^m(x,t)e^{pv(0,t)}$, $v_t = \Delta v + u^q(0,t)e^{nv(x,t)}$, subject to homogeneous Dirichlet boundary conditions. The complete classification on non-simultaneous and simultaneous blow-up is obtained by four sufficient and necessary conditions. It is interesting that, in some exponent region, large initial data $u_0(v_0)$ leads to the blow-up of $u(v)$, and in some betweenness, simultaneous blow-up occurs. For all of the nonnegative exponents, we find that $u(v)$ blows up only at a single point if $m > 1 (n > 0)$, while $u(v)$ blows up everywhere for $0 \leqslant m \leqslant 1$ $(n = 0)$. Moreover, blow-up rates are considered for both non-simultaneous and simultaneous blow-up solutions.

1. Introduction

In the present paper, we consider the heat equations

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u^m(x,t)e^{pv(0,t)}, & (x,t) \in \Omega \times (0,T), \\ v_t(x,t) = \Delta v(x,t) + u^q(0,t)e^{nv(x,t)}, & (x,t) \in \Omega \times (0,T), \end{cases} \tag{1.1}$$

subject to homogeneous Dirichlet boundary conditions, with initial data $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, $x \in \Omega$, where $\Omega = B_R = \{|x| < R\} \subset \mathbb{R}^N$; exponents m, n, p, and q are nonnegative; T represents the maximal existence time of the solutions; initial data $u_0, v_0 : \overline{B}_R \to \mathbb{R}^1$ are nonnegative, nontrivial, radially symmetric non-increasing continuous functions, vanishing on ∂B_R. The existence and uniqueness of local classical solutions to (1.1) is well known (see, for example, [9]). Non-linear parabolic systems like (1.1) come from population dynamics, chemical reactions, heat transfer, etc., where u and v represent the densities of two biological populations during a migration, the thickness of two kinds of chemical reactants, the temperatures of two different materials during a propagation, etc.

Chen [3] discussed the following heat equations

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u^m(x,t)v^p(x,t), & (x,t) \in \Omega \times (0,T), \\ v_t(x,t) = \Delta v(x,t) + u^q(x,t)v^n(x,t), & (x,t) \in \Omega \times (0,T), \end{cases} \tag{1.2}$$

subject to null Dirichlet boundary conditions, where $q > m - 1$ and $p > n - 1$, Ω is a general bounded domain of $\mathbb{R}^N$. He proved that, if $m, n \leqslant 1$, and $pq \leqslant (1-m)(1-n)$, then all nonnegative solutions are global, while both global and blow-up solutions coexist if $m > 1$, or $n > 1$, or $pq > (1-m)(1-n)$, where the blow-up for nonnegative solutions is defined as $\limsup_{t\to T}\left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)}\right) = +\infty$. We say that the solution (u,v) blows up simultaneously if $\limsup_{t\to T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \limsup_{t\to T}\|v(\cdot,t)\|_{L^\infty(\Omega)} = +\infty$. So the non-simultaneous blow-up means that, e.g., $\limsup_{t\to T}$

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$\|u(\cdot,t)\|_{L^\infty(\Omega)} = +\infty$ with $\|v(\cdot,t)\|_{L^\infty(\Omega)} < +\infty, t \in [0,T]$. The simultaneous blow-up rate of (1.2) in some exponent region was obtained by Wang [21] and Zheng [23] for radially symmetric solutions in B_R as follows,

$$\max_{\overline{B}_R} u(\cdot,t) \sim (T-t)^{-\frac{p+1-n}{pq-(1-m)(1-n)}}, \quad \max_{\overline{B}_R} v(\cdot,t) \sim (T-t)^{-\frac{q+1-m}{pq-(1-m)(1-n)}}.$$

The other studies about system (1.2) were considered in e.g., [5,7,19,20], where the blow-up criteria, blow-up rate, and even blow-up profile were considered.

The non-simultaneous blow-up had been observed and discussed by Quirós and Rossi [15] for the Cauchy problem of (1.2) in $\mathbb{R}^N$. They proved that there exists initial data such that u blows up while v remains bounded if $m > q + 1$; If u blows up at some point $x_0 \in \mathbb{R}^N$ while v remains bounded and

$$u(x,t) \geqslant c(T-t)^{-\frac{1}{m-1}}, \quad |x-x_0| \leqslant K\sqrt{T-t}, \tag{1.3}$$

then $m > q + 1$. It was noted that the results of [15] also hold for system (1.2). The restriction condition (1.3) had been removed by Brändle et al. [2] for the Cauchy problem of (1.2) with $N = 1$. There are also some results for local non-linearities with respect to the phenomena of non-simultaneous blow-up (see, for example, [10,15,16,18]).

Recently, Li and Wang [11] considered the following non-linear parabolic system

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u^m(x,t)v^p(x_0,t), & (x,t) \in \Omega \times (0,T), \\ v_t(x,t) = \Delta v(x,t) + u^q(x_0,t)v^n(x,t), & (x,t) \in \Omega \times (0,T), \end{cases} \tag{1.4}$$

under null Dirichlet boundary conditions, where exponents $m, n, p, q \geqslant 0$, $m + p > 0$, $n + q > 0$. The results can be summarized as follows,

- $m, n \leqslant 1$: The blow-up classical solution (u, v) is simultaneous, and possesses the total blow-up pattern. Moreover, the uniform blow-up profiles are obtained.
- $m, n > 1$ with $x_0 = 0$, $\Omega = B_R = \{|x| < R\}$: Assume that $u_0, v_0 : \overline{B}_R \to \mathbb{R}^1$ are nonnegative nontrivial, radially symmetric non-increasing continuous functions and vanish on ∂B_R, and $\Delta u_0(x) + u_0^m(x)v_0^p(0) \geqslant 0, \Delta v_0(x) + u_0^q(0)v_0^n(x) \geqslant 0, x \in B_R$. The following results were obtained:
  - A sufficient condition for only simultaneous blow-up: If $q \geqslant m-1 > 0$ and $p \geqslant n-1 > 0$, then any blow-up must be simultaneous.
  - A necessary condition for the existence of simultaneous blow-up: If simultaneous blow-up happens, then the exponent regions must be $q \geqslant m-1 > 0$ and $p \geqslant n-1 > 0$, or $q < m-1$ and $p < n-1$. In addition, the simultaneous blow-up rates were obtained. (u, v) blows up only at $x = 0$.
  - The blow-up rates in space are evaluated: $u(r,t) \leqslant Cr^{-\alpha}$, $v(r,t) \leqslant Cr^{-\beta}$, $(r,t) \in (0,R] \times [0,T)$ with $\alpha > 2/(m-1)$ and $\beta > 2/(n-1)$, and $u_0'(r) \leqslant -cr$, $v_0'(r) \leqslant -cr$ in $[0,R]$.

Zheng et al. [24] considered the radially symmetric solutions of the following homogeneous Dirichlet problem

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + e^{mu(x,t)+pv(x,t)}, & (x,t) \in B_R \times (0,T), \\ v_t(x,t) = \Delta v(x,t) + e^{qu(x,t)+nv(x,t)}, & (x,t) \in B_R \times (0,T). \end{cases} \tag{1.5}$$

They obtained the solutions blow-up only at $x = 0$ with simultaneous blow-up rate

$$e^{u(0,t)} \sim (T-t)^{-\frac{p-n}{pq-mn}}, \quad e^{v(0,t)} \sim (T-t)^{-\frac{q-m}{pq-mn}}$$

in the region $q > m \geqslant 0$, $p > n \geqslant 0$. The other known results to special cases of (1.5) were obtained in [7,8,13,14,22], etc. For the surveys on non-linear parabolic systems, one can refer to [1,4].

Motivated by the above works, we discuss the classification on non-simultaneous versus simultaneous blow-up, blow-up rates and blow-up sets of system (1.1) in the present paper, which had not been considered before. It can be checked by the above results that, if $m > 1$, or $n > 0$, or $pq > n(m-1)$, then solutions of (1.1) blow-up for large initial data. In the sequel, we always consider blow-up solutions of the system and assume that initial data satisfies

$$\Delta u_0(x) + (1-\varepsilon)u_0^m(x)e^{pv_0(0)} \geqslant 0, \quad \Delta v_0(x) + (1-\varepsilon\varphi(x))u_0^q(0)e^{nv_0(x)} \geqslant 0, \quad x \in B_R,$$

where positive ε is small and $\varphi(x) > 0$ is the first eigenfunction of $(-\Delta)$ in B_R with homogeneous Dirichlet boundary condition, normalized by $\|\varphi\|_\infty = 1$. For fixed constant $\varepsilon \in (0,1)$, define $\mathbb{V}_0$ as a set of initial data above. It is easy to check that u_t, $v_t \geqslant 0$ by the comparison principle.

The present paper is arranged as follows. In the next section, we give the optimal and complete classifications on non-simultaneous and simultaneous blow-up solutions. In Section 3, all kinds of blow-up sets are proved also with blow-up rates estimates.