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Exact and explicit solutions of Euler–Painlevé equations through generalized Cole–Hopf transformations

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ABSTRACT

More general Euler–Painlevé equations are exactly linearized using generalized Cole–Hopf transform and are shown to admit exact solutions in terms of Kummer functions. The asymptotic behaviours of Euler–Painlevé equations are also derived.

1. Introduction

Sachdev and his collaborators [9–11] have introduced Euler–Painlevé equations (EPEs) as self-similar reductions of generalized Burges equations (GBEs). They have found approximate analytical solutions by discarding the nonlinear terms in the EPEs.

Recently Mayil Vaganan and Senthilkumaran [6] have found exact, explicit solutions of EPEs in term of Kummer functions by exactly linearizing the EPE using a generalization of Cole–Hopf [3,4] transformation. Indeed they linearized the EPE

$$HH'' - 2H'^2 + \frac{2c_3}{r_1}HH' + 2\frac{\alpha c_0 + c_3}{r_1}H^2 - \frac{2}{r_1}H' + \frac{2A_0B_0\alpha^2}{r_1}zH^3 = 0, \tag{1.1}$$

to the Kummer equation

$$\zeta g''(\zeta) + \left(\frac{1}{2} - \zeta\right)g'(\zeta) - \left[\frac{2ka_1 + c_1}{ka_1 + 2\alpha a_2}\right]g(\zeta) = 0, \tag{1.2}$$

through a generalization of Cole–Hopf transformation

$$H(z) = \frac{2g(\zeta)}{z(4ka_1 g(\zeta) - (2\alpha a_2 + ka_1)g'(\zeta))}, \tag{1.3}$$

$$\zeta(z) = \left(\frac{a_1 k + 2\alpha a_2}{4a_1}\right)z^2, \tag{1.4}$$

where c_1 is the constant of integration.

Generally EPEs do not have solutions expressible in terms of known functions. EPEs seem to be analytically much nicer than the Painlevé equations and in the physically interesting cases do not exhibit any singularities. The scope of the EPEs is considerably enlarged by juxtaposing them with a large number nonlinear differential equations acquired by Kamke [5] and

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Murphy [7] from different sources and applications. It would seem, therefore, that the EPEs have a larger role to play in a variety of physical applications than would be suggested by generalized Burgers equations alone.

The purpose of the present paper is to determine exact, explicit solutions for a few more EPEs using the same technique of exact linearization through generalized Cole–Hopf transformations. The EPEs to be studied here are

$$aHH'' - 2aH'^2 - \alpha a_2 zHH' - 2H' + \alpha a_2 H^2 = 0, \tag{1.5}$$

$$HH'' - 2H'^2 - \alpha a_2 zHH' - 2H' + \alpha a_2 H^2 + 2a_3 a_4 \alpha^2 zH^3 = 0, \tag{1.6}$$

$$HH'' - 2H'^2 + 2cHH' - 2H' = 0, \tag{1.7}$$

$$HH'' - 2H'^2 + 2a_1 a_2 zHH' - 2H' - 2a_1 a_2 H^2 - 2a_1^2 a_2^2 zH^3 = 0, \tag{1.8}$$

$$HH'' - \frac{n+1}{n}H'^2 - \frac{2}{nc_2 r_2}H' - \frac{2}{nc_2 c_0 r_2}zHH' + \left(\frac{jnc_0 + 2nc_2 - 2c_0}{n^2 c_0 c_2^2 r_2}\right)H^2 = 0, \tag{1.9}$$

which are obtained as similarity reductions by applying the Lie's classical method to the GBEs

$$u_t + uu_x + \alpha u = \frac{ae^{-\alpha t}}{2}u_{xx}, \tag{1.10}$$

$$u_t + uu_x = \frac{1}{2}\left(\frac{a_1 t + a_4}{a_2 t + a_3}\right)^r u_{xx}, \tag{1.11}$$

$$u_t + u^n u_x + \frac{ju}{2t} = \frac{t}{2}u_{xx}. \tag{1.12}$$

The scheme of the paper is as follows: Section 2 deals with the exact solution of the EPEs (1.5)–(1.9). The conclusion of the present work is set forth in Section 3.

2. Exact solutions of Euler–Painlevé equations

We organise this section into three Subsections 2.1–2.3. In 2.1, we work with the EPEs (1.5)–(1.7). Sections 2.2 and 2.3 respectively deals with (1.8) and (1.9).

2.1. EPEs (1.5)–(1.7)

An application of Lie's classical method [2,8] to (1.10) results in the following three reductions:

Reduction-1:

$$u = \left[f(z) - \alpha a_4 e^{-\alpha t} z\right] e^{-\frac{\alpha t}{2}} \left(a_4 e^{-\alpha t} + a_2\right)^{-1/2}, \tag{2.1}$$

$$z = xe^{\frac{\alpha t}{2}}\left(a_4 e^{-\alpha t} + a_2\right)^{-1/2}, \tag{2.2}$$

where a_2, a_4 are real constants. Using (2.1) and (2.2) in (1.10), we obtain

$$af'' - \alpha a_2 zf' - 2ff' - \alpha a_2 f = 0. \tag{2.3}$$

Now the transformation $f = H^{-1}$ replaces (2.3) by the EPE

$$aHH'' - 2aH'^2 - \alpha a_2 zHH' - 2H' + \alpha a_2 H^2 = 0. \tag{2.4}$$

Integration of (2.4) gives

$$aH' + \alpha a_2 zH + 1 - c_1 H^2 = 0, \tag{2.5}$$

where c_1 is constant of integration.

Now using the transformation

$$H(z) = \frac{1}{-\alpha a_2}\frac{1}{z}\left(1 + \frac{3}{4}\frac{g'(\zeta)}{g(\zeta)}\right)^{-1}, \quad \zeta = \frac{3\alpha a_2}{8a}z^2, \tag{2.6}$$

we replace (2.5) by the Kummer equation for $g(\zeta)$, viz.,

$$\zeta g''(\zeta) + \left(\frac{1}{2} - \zeta\right)g'(\zeta) - \left[\frac{2c_1}{3\alpha a_2} - \frac{2}{3}\right]g(\zeta) = 0. \tag{2.7}$$

Two linearly independent solutions of Eq. (2.7) are

$$g_1(\zeta) = M\left(\frac{2c_1}{3\alpha a_2} - \frac{2}{3}, \frac{1}{2}; \zeta\right), \quad g_2(\zeta) = U\left(\frac{2c_1}{3\alpha a_2} - \frac{2}{3}, \frac{1}{2}; \zeta\right), \tag{2.8}$$

where M and U are Kummer functions.