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Quasilinear iterative scheme for a fourth-order differential equation with retardation and anticipation ☆

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ABSTRACT

In this paper, the estimations of error between the approximate solution and the solution for a fourth-order differential equation with retardation and anticipation are given by employing the quasilinear iterative scheme and the approximate quasilinear iterative scheme.

1. Introduction

In this paper, we consider the following boundary value problem of a fourth-order differential-difference equation

$$\begin{cases} x^{(4)}(t) - c_1 x^{(4)}(t-\tau_1) - c_2 x^{(4)}(t+\tau_2) = f(t;\bar{x}(t),\bar{x}(t-\tau_1),\bar{x}(t+\tau_2)), \quad a<t<b, \\ x(t)=0, \quad t\in[a-\tau_1,a]\cup[b,b+\tau_2], \\ x'(a+0)=0, \quad x'(b-0)=0, \end{cases} \tag{1.1}$$

where $\bar{x}(t) = (x(t),x'(t),x''(t),x'''(t))$, $\bar{x}(t+\Delta) = (x(t+\Delta),x'(t+\Delta),x''(t+\Delta),x'''(t+\Delta))$, $(\Delta = -\tau_1,\tau_2)$. τ_1,τ_2 are positive numbers, and assume $|c_1| + |c_2| < 1$, f is continuous in t.

Functional differential equation with both retardation and anticipation provide a mathematical model for the current evolution process of such systems. This study was propelled by the fact that specific equations with retardation and anticipation appear in modeling [1,2]. Recently, there are some results of existence theory and quadratic convergence on such equations, we can refer to the monograph of Lakshmikantham and Vatsala [3], papers of Wang and Li [4], Wang [5,6], Bhaskar and Lakshmikantham [7], Bernfeld et al. [8], Bhaskar et al. [9], Dricia et al. [10] and references cited therein. In general, the analytical solution of such equations are difficult to obtain. The problem of approximation of solutions has attracted a lot of attention during the past decades. For example, Agarwal and Chow [11], Sun and Wang [12] gave Picard's iterative results and the approximate Picard's iterative results for a four order ordinary differential equations and a four order functional differential equations with retardation and anticipation respectively. However, to the best of our knowledge, there are few results on the iterative and approximate iterative solution, especially the estimation of error between the approximate solution and the solution. Motivated by Agarwal and Chow [11], We shall, in this paper, discuss the estimations of the error between the approximate solution and the solution are given by the quasilinear iterative scheme and the approximate quasilinear iterative scheme.

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2. Preliminaries

In order to state our results, we need some notions and Lemmas which are important to our results.

Definition 2.1. A function $x(t)$ is called a solution of the BVP (1.1). If (I) $x(t) \in C^{(3)}[a-\tau_1, b+\tau_2]$, (II) $x(t)$ satisfies boundary condition, (III) Eq. (1.1) holds for $t \in [a,b]$, $t \neq a+k\tau_1$, $t \neq b-k\tau_2 (k \in Z^+)$.

Lemma 2.1 (*see [6]*). *Let $x(t) \in K = \{x(t)$: $x^{(4)}(t)$ on $[a,b]$ has only some limited discontinuous points of the first kind, $x(t) = 0$, $t \in [a-\tau_1, a] \cup [b, b+\tau_2]$, $x'(a+0) = x'(b-0) = 0\}$, then*

$$|x^{(k)}(t)| \leqslant C_{4,k}(b-a)^{4-k}M, \quad a-\tau_1 \leqslant t \leqslant b+\tau_2, \ k = 0,1,2,3, \tag{2.1}$$

where $M = \max_{a \leqslant t \leqslant b}|x^{(4)}(t)|, \ C_{4,0} = \frac{1}{384}, \ C_{4,1} = \frac{1}{72\sqrt{3}}, \ C_{4,2} = \frac{1}{12}, \ C_{4,3} = \frac{1}{2}$.

Lemma 2.2 (*Toeplitz Lemma*). *For any $0 \leqslant \alpha < 1$, let $S_m = \sum_{i=0}^{m} \alpha^{m-i} d_i, \ d_i \in R, \ i = 0,1,2,\ldots,m$. Then*

$$\lim_{m\to\infty} S_m = 0 \quad \text{if and only if} \quad \lim_{m\to\infty} d_m = 0. \tag{2.2}$$

Define the Green's function as follows:

$$G(t,s) = \frac{1}{6(b-a)^2}\begin{cases} (t-a)^2(b-s)^2\left((s-t) + \frac{2(b-t)(s-a)}{(b-a)}\right), & a \leqslant t \leqslant s \leqslant b, \\ (s-a)^2(b-t)^2\left((t-s) + \frac{2(t-a)(b-s)}{(b-a)}\right), & a \leqslant s \leqslant t \leqslant b, \\ 0, \quad a-\tau_1 \leqslant s < a \text{ or } b < s \leqslant b+\tau_2. \end{cases}$$

Definition 2.2. A function $z(t)$ is called an approximate solution of the BVP (1.1), if there exists a constant $\varepsilon > 0$ such that

$$\max_{a \leqslant t \leqslant b}|z^{(4)}(t) - c_1 z^{(4)}(t-\tau_1) - c_2 z^{(4)}(t+\tau_2) - f(t;\bar{z}(t),\bar{z}(t-\tau_1),\bar{z}(t+\tau_2))| < \varepsilon, \tag{2.3}$$

and

$$z(t) = 0, \quad t \in [a-\tau_1, a] \cup [b, b+\tau_2]; \quad z'(a+0) = 0, \quad z'(b-0) = 0.$$

In fact, the approximate solution $z(t)$ can be expressed as:

$$z(t) = \int_{a-\tau_1}^{b+\tau_2} G(t,s)\{f(s;\bar{z}(s),\bar{z}(s-\tau_1),\bar{z}(s+\tau_2)) + c_1 z^{(4)}(s-\tau_1) + c_2 z^{(4)}(s+\tau_2) + \eta(s)\}ds, \tag{2.4}$$

in which

$$\eta(t) = z^{(4)}(t) - c_1 z^{(4)}(t-\tau_1) - c_2 z^{(4)}(t+\tau_2) - f(t;\bar{z}(t),\bar{z}(t-\tau_1),\bar{z}(t+\tau_2)), \quad \max_{a \leqslant t \leqslant b}|\eta(t)| \leqslant \varepsilon.$$

Definition 2.3. The function f is said to be of the Lipschitz class, if for all $(t;\bar{u}(t),\bar{u}(t-\tau_1),\bar{u}(t+\tau_2))$, $(t;\bar{v}(t), \bar{v}(t-\tau_1), \bar{v}(t+\tau_2)) \in [a,b] \times D, \ D \subset R^{12}$, the following inequality is satisfied

$$\begin{aligned} &|f(t;\bar{u}(t),\bar{u}(t-\tau_1),\bar{u}(t+\tau_2)) - f(t;\bar{v}(t),\bar{v}(t-\tau_1),\bar{v}(t+\tau_2))| \\ &\quad \leqslant sum_{j=0}^{3}\{h_{j+1}|u^{(j)}(t) - v^{(j)}(t)| + l_{j+1}|u^{(j)}(t-\tau_1)| - v^{(j)}(t-\tau_1) + m_{j+1}|u^{(j)}(t+\tau_2) - v^{(j)}(t+\tau_2)|\}, \end{aligned} \tag{2.5}$$

in which h_{j+1}, l_{j+1}, m_{j+1} and N are nonnegative constants, and

$$D = \left\{(\bar{x}(t),\bar{x}(t-\tau_1),\bar{x}(t+\tau_2)) : |x^{(j)}(t) - z^{(j)}(t)| \leqslant \frac{NC_{4,j}}{C_{4,0}(b-a)^j}, \quad |x^{(j)}(t+\Delta) - z^{(j)}(t-\Delta)| \leqslant \frac{NC_{4,j}}{C_{4,0}(b-a)^j}, \Delta = -\tau_1, \tau_2, \quad 0 \leqslant j \leqslant 3\right\}.$$

Denote: $R_4 = \{x(t)$: $x(t) \in C^{(3)}[a-\tau_1, b+\tau_2]$; $x(t) = 0$, $t \in [a-\tau_1, a] \cup [b, b+\tau_2]$, $x'(a+0) = x'(b-0) = 0$; $x^{(4)}(t)$ on $[a,b]$ has only some limited discontinuous points of the first kind$\}$.

For $x(t) \in R_4$, we define the norm

$$\|x\| = \max_{0 \leqslant j \leqslant 3}\left\{\frac{C_{4,0}(b-a)^j}{C_{4,j}}\max_{a \leqslant t \leqslant b}|x^{(j)}(t)|, \frac{C_{4,0}(b-a)^j}{C_{4,j}}\max_{a \leqslant t \leqslant b}|x^{(j)}(t+\Delta)|, \Delta = -\tau_1, \tau_2\right\}.$$

It is easy to verify R_4 is a Banach space and we can obtain the following Lemma by simple calculus.