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Superconvergence of a discontinuous finite element method for a nonlinear ordinary differential equation

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ABSTRACT

In this paper, n-degree discontinuous finite element method with interpolated coefficients for an initial value problem of nonlinear ordinary differential equation is introduced and analyzed. By using the finite element projection for an auxiliary linear problem as comparison function, an optimal superconvergence $u - U = \mathcal{O}(h^{n+2}),\ n \geqslant 2$, at $(n+1)$-order characteristic points in each element respectively is proved. Finally the theoretic results are tested by a numerical example.

1. Introduction

Consider an initial value problem of nonlinear ordinary differential equation

$$u' = f(t,u), \quad t \in (0,T],\ u(0) = u_0, \tag{1}$$

where $f(t,u)$ is a sufficiently smooth function. Let $\mathcal{J}_h$ be a partition of $I = [0,T]$ such that $\mathcal{J}_h : 0 = t_0 < t_1 < \cdots < t_N = T$, set element $I_j = (t_{j-1}, t_j)$, midpoint $t_{j-1/2} = (t_j + t_{j-1})/2$ and half-step $h_j = (t_j - t_{j-1})/2$, $h = \max(h_j)$, $j = 1,2,\ldots,N$. Assume that $\mathcal{J}_h$ is quasi-uniform, i.e., there is a constant $C > 0$ such that $h \leqslant Ch_j$. For the partition $\mathcal{J}_h$ define the finite element space to be

$$S^h = \{u : u|_{I_j} \in \mathbf{P}_n(I_j),\ j = 1,2,\ldots,N,\ u(t_j) = u(t_j - 0),\ j = 0,1,2,\ldots,N\},$$

where $\mathbf{P}_n(I_j)$ denotes the space of all univariable polynomials of degree $\leqslant n$ in I_j. On the element I_j, an n-degree polynomial has $n+1$ parameters, so the finite element on this element has $n+1$ freedom degree. Classical discontinuous finite element solution $\overline{U} \in S^h$ of (1) satisfies

$$\int_{I_j} (\overline{U}' - f(t,\overline{U}))v\,\mathrm{d}t + [\overline{U}_{j-1}]v^+_{j-1} = 0, \quad v \in S^h,\ j = 1,2,\ldots,N. \tag{2}$$

For the sake of simplicity, we now define n-degree discontinuous finite element with interpolated coefficients, $U \in S^h$, by

$$\int_{I_j} (U' - I_h f(t,U))v\,\mathrm{d}t + [U_{j-1}]v^+_{j-1} = 0, \quad v \in S^h,\ j = 1,2,\ldots,N, \tag{3}$$

where I_h denotes the Lagrangian interpolating operator on S^h, and U and $I_h f(t,U)$ satisfy

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$$U = \sum \psi_\alpha(t)U(t_\alpha), \quad I_h f(t,U) = \sum \psi_\alpha(t) f(t_\alpha, U(t_\alpha)),$$

where $\psi_\alpha(t)$ are the basis functions in the element I_j. Note that the exact solution of (1) satisfies, for smooth function v,

$$\int_{I_j} (u' - f(t,u))v\,dt = 0, \quad j = 1,2,\ldots,N \tag{4}$$

and hence, subtracting (2) from (4), we have

$$\int_{I_j} (e' - f(t,u) + I_h f(t,U))v\,dt + [e_{j-1}]v_{j-1}^+ = 0, \quad v \in S^h, \; j = 1,2,\ldots,N, \tag{5}$$

where $e = u - U$ with $e(0) = u_0 - U_0 = u_0 - U_0^- = 0$.

In 1974, Lesaint–Raviart introduced the discontinuous finite element method (DFEM) [6]. Delfour–Hager–Trocher (1981) presented the discontinuous finite element of the ordinary differential problem and its superconvergence [3]. Eriksson–Johnson–Thomee [4] and Thomee [8] studied the discontinuous finite element method for the heat problem. Chen proved superconvergence of DFEM for linear case $f(t,u) = au + b$ by a new element orthogonality analysis [1]. In virtue of a simple argument Li–Chen obtained superconvergence of classical DFEM for nonlinear problem (1) [7]. The finite element method with interpolated coefficients was introduced and analyzed for semilinear parabolic problems in Zlamal et al. [13]. Later Larsson–Thomee–Zhang studied the semidiscrete linear triangular finite element and obtained an error estimate [5]. Chen–Larsson–Zhang derived almost optimal order convergence on piecewise uniform triangular meshes by use of superconvergence techniques [2]. Xiong–Chen studied the superconvergence of the finite element with interpolated coefficients for semilinear elliptic problem and the continuous finite element with interpolated coefficients for an initial value problem of nonlinear ordinary differential equation [9–11]. Recently Xiong–Chen–Zhang studied the superconvergence of the semidiscrete finite element with interpolated coefficients for semilinear hyperbolic equation [12]. By the compendious argument we shall study superconvergence of discontinuous finite element with interpolated coefficients for an initial value problem of the nonlinear ordinary differential equation (1). Finally the theoretical results are tested by a numerical example.

For our analysis, we introduce the Legendre's polynomials in the interval $E = (-1,1)$

$$l_0 = 1, l_1 = s, l_2 = \frac{1}{2}(3s^2 - 1), \ldots, l_n = \frac{1}{2^n n!}\frac{d^n}{ds^n}(s^2 - 1)^n, \ldots, \tag{6}$$

where the inner product $(l_i, l_j) = 0$ if $i \neq j$, otherwise $(l_i, l_j) = \frac{2}{2j+1}, l(\pm 1) = (\pm 1)^j$. Introduce [1] the basic functions

$$\varphi_n = l_n(s) - l_{n-1}(s), \quad n > 0, \;\; \varphi_0 = 1, \tag{7}$$

which has the property $(\varphi_n, \psi) = 0$, $\psi \in \mathbf{P}_{n-2}(E)$, $n \geqslant 2$. The polynomial φ_{n+1} has $n + 1$ distinct roots ($(n + 1)$ order characteristic points): $-1 < z_1 < z_2 < \cdots < z_{n+1} = 1$ in $\bar{E} = [-1, 1]$. Denotes set of $(n + 1)$-order characteristic points in all elements in the partition $\mathcal{J}_h$ by

$$Z^* = \{t_{ji} = t_{j-1/2} + h_j z_i, \; j = 1,2,\ldots,N, \; i = 1,2,\ldots,n+1\}.$$

Here and below, denote Sobolev space and its norm by $W^{k,p}(I)$ and $\|u\|_{k,p,I}$, respectively. If $p = 2$, simply use $H^k(I)$ and $\|u\|_{k,I}$.

Our main result about the discontinuous finite element with interpolated coefficients for the initial value problem of the nonlinear ordinary differential equation is the following.

Theorem 1.1. *Assume that the partition $\mathcal{J}_h$ of the interval $I = [0,T]$ is quasiuniform and let $U \in S^h$ be n-degree discontinuous finite element solution with interpolated coefficients of (1). Then, at $z \in Z^*$, there is superconvergence estimate*

$$(u - U)(z) = \mathcal{O}(h^{n+2}), \quad n \geqslant 2. \tag{8}$$

2. Proof of theorem

First denote $a = a(t)$ by $f_u(t, U(t))$ where $U = U(t)$ is the solution of (3), then consider the finite element projection for an auxiliary linear problem, $\widetilde{U} \in S^h$, such that

$$\int_{I_j} ((u' - \widetilde{U}') - a(u - \widetilde{U}))v\,dt + [(u - \widetilde{U})_{j-1}]v_{j-1}^+ = 0, \quad v \in S^h, \; j = 1,2,\ldots,N, \tag{9}$$

with $\widetilde{U}_0 = \widetilde{U}_0^- = u_0$. Recalling the result of the discontinuous finite element for linear problem of ordinary differential equation [1], we obtain following superconvergence estimate.

Lemma 2.1. *Let $I_h u$ and $\widetilde{U}$ be the interpolation of exact solution u of (1) and auxiliary linear projection defined by (9), respectively. Then at $z \in Z^*$, there is superconvergence estimate*