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Existence of analytic solutions of an iterative functional equation ☆

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ABSTRACT

This paper is concern analytic solutions of an iterative functional equation of the form

$$f(p(z) + q(f(z))) = h(f(z)), z \in \mathbb{C}.$$

Byconstructing a convergent power series solution of an auxiliary equation

$$g(\phi(\alpha^2 z) - p(\phi(\alpha z))) = h(g(\phi(\alpha z) - p(\phi(z)))), z \in \mathbb{C},$$

analytic solutions of the original equation are obtained. We discuss not only these α appeared in the auxiliary equation at the hyperbolic case $0 < |\alpha| \neq 1$ and resonance, i.e., at a root of the unity, but also those α near resonance (i.e., near a root of the unity) under Brjuno condition.

1. Introduction

Invariant curves of the area preserving maps play an important role in the theory of periodic stability of discrete dynamical systems. A common and useful method to understand behaviors of a discrete dynamical system generated by the iteration of a self-mapping is to find a simple structures in its phase space and to describe the dynamics in terms of the effect caused by the presence of these structures. In particular, invariant curve, in dimension two, is one of such structures. Diamond [1] researched the existence of analytic invariant curves for two-dimensional maps of the form

$$T(x, y) = (x + y, y(1 + \beta x^k) + F(x, y)).$$

For an investigation of other related problems the interested reader is referred to [2–12] and the monograph [13]. In particular, Wen Rong Li and Sui Sun Cheng [4] discussed the analytic solutions of functional equation

$$f(p(z) + bf(z)) = h(z) \tag{1.1}$$

by using the method of majorant series. The purpose of this paper is to find invertible and linearizable invariant analytic curves of the 2-D complex map $T : \mathbb{C}^2 \to \mathbb{C}^2,\ (z, w) \mapsto (z_1, w_1)$, defined by

$$\begin{cases} z_1 = p(z) + q(w), \\ w_1 = h(w). \end{cases} \tag{1.2}$$

Throughout this paper, we assume that $p(z)$, $q(z)$ and $h(z)$ are analytic in a neighborhood of the origin, and $p(0) = 0, h(0) = 0, q(0) = 0, p'(0) = \xi \neq 0, h'(0) = \eta \neq 0$ and $q'(0) = \zeta \neq 0$. Clearly, map T has a fixed point $O = (0,0)$ with the Jacobian matrix

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$$A = DT(0) = \begin{pmatrix} \xi & \zeta \\ 0 & \eta \end{pmatrix}$$

at O. Its characteristic polynomial is

$$P_A(\lambda) = \alpha^2 - (\xi + \eta)\alpha + \xi\eta.$$

As is well known that a curve $w = f(z)$ is said to be an invariant curve of T if $w_1 = f(z_1)$. Thus, we see that map T has an invariant curve $w = f(z)$ if and only if f satisfies the functional equation

$$f(p(z) + q(f(z))) = h(f(z)), \ z \in \mathbb{C}. \tag{1.3}$$

The results of this paper can be regarded as a generalization of the results obtained in [4].

From the above assumptions, it is easy to see that the inverse function of $q(z)$ exists and is analytic in a neighborhood of the $q(0) = 0$. We denote the inverse function of $q(z)$ by $g(z)$ and consider the following equation

$$g(\phi(\alpha^2 z) - p(\phi(\alpha z))) = h(g(\phi(\alpha z) - p(\phi(z)))), \ z \in \mathbb{C}, \tag{1.4}$$

which is called the auxiliary equation of (1.3). We first construct analytic solutions of (1.4) in the cases:

- **(H1)** $0 < |\alpha| \neq 1$.
- **(H2)** $\alpha = e^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is a Brjuno number ([14] and [15]), i.e., $B(\theta) = \sum_{k=0}^{\infty} \frac{\log q_{k+1}}{q_k} < \infty$, where $\{p_k/q_k\}$ denotes the sequence of partial fraction of the continued fraction expansion of θ, said to satisfy the *Brjuno condition*.
- **(H3)** $\alpha = e^{2\pi i\, q/p}$ for some integers $p \in \mathbb{N}$ with $p \geqslant 2$ and $q \in \mathbb{Z} \setminus \{0\}$, and $\alpha \neq e^{2\pi i l/k}$ for all $1 \leqslant k \leqslant p-1$ and $l \in \mathbb{Z} \setminus \{0\}$.

Observe that α is the inside or outside of the unit circle S^1 in the case of (H1) but on S^1 in the rest cases. More difficulties are encountered for α on S^1, as mentioned in the so-called "small-divisor problem" (seen in [16] p. 22 and p. 146 and [17]). Under Diophantine condition: $\alpha = e^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and there exist constants $\zeta > 0$ and $\sigma > 0$ such that $|\alpha^n - 1| \geqslant \zeta^{-1} n^{-\sigma}$ for all $n \geqslant 1$, the number $\alpha \in S^1$ is "far" from all roots of the unity and was considered in different settings [10–12]. In [10] the case of (H3), where α is a root of the unity, was also discussed for a general class of iterative equations. Since then, we have been striving to give a result of analytic solutions for those α "near" a root of the unity, i.e., neither being roots of the unity nor satisfying the Diophantine condition.

The Brjuno condition in (H2) provides such a chance for us. As stated in [18], for a real number θ, we let $[\theta]$ denote its integer part and $\{\theta\} = \theta - [\theta]$ its fractional part. Then every irrational number θ has a unique expression of the Gauss' continued fraction

$$\theta = a_0 + \theta_0 = a_0 + \frac{1}{a_1 + \theta_1} = \cdots,$$

denoted simply by $\theta = [a_0, a_1, \ldots, a_n, \ldots]$, where a_j's and θ_j's are calculated by the algorithm: (**a**) $a_0 = [\theta]$, $\theta_0 = \{\theta\}$, and (**b**) $a_n = [\frac{1}{\theta_{n-1}}]$, $\theta_n = \{\frac{1}{\theta_{n-1}}\}$ for all $n \geqslant 1$. Define the sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as follows:

$$q_{-2} = 1, \ q_{-1} = 0, \ q_n = a_n q_{n-1} + q_{n-2};$$
$$p_{-2} = 0, \ p_{-1} = 1, \ p_n = a_n p_{n-1} + p_{n-2}.$$

It is easy to show that $p_n/q_n = [a_0, a_1, \ldots, a_n]$. Thus, For every $\theta \in \mathbb{R} \setminus \mathbb{Q}$ we associate, using its convergence, an arithmetical function $B(\theta) = \sum_{n\geqslant 0} \frac{\log q_{n+1}}{q_n}$. We say that θ is a Brjuno number or that it satisfies Brjuno condition if $B(\theta) < +\infty$. The Brjuno condition is weaker than the Diophantine condition. For example, if $a_{n+1} \leqslant c e^{a_n}$ for all $n \geqslant 0$, where $c > 0$ is a constant, then $\theta = [a_0, a_1, \ldots, a_n, \ldots]$ is a Brjuno number but is not a Diophantine number. So, the case (H2) contains both Diophantine condition and a part of α "near" resonance.

In this paper, considering the Brjuno condition instead of the Diophantine one, we discuss not only the cases (H1) and (H3) but also (H2) for analytic invariant curves of the mapping T defined in (1.2).

2. Substituting power series in Eq. (1.4)

Let

$$p(z) = \sum_{n=1}^{\infty} a_n z^n, \quad h(z) = \sum_{n=1}^{\infty} c_n z^n, \quad g(z) = \sum_{n=1}^{\infty} g_n z^n, \quad a_1 = \xi, \ c_1 = \eta, \ g_1 = s = \frac{1}{\zeta}. \tag{2.1}$$

Without loss of generality , we can assume that

$$|a_n| \leqslant 1, \ |c_n| \leqslant 1, \ |g_n| \leqslant 1, \ n = 2, 3, \ldots. \tag{2.2}$$