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Numerical continuation methods for studying periodic travelling wave (wavetrain) solutions of partial differential equations

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ABSTRACT

Periodic travelling waves (wavetrains) are an important solution type for many partial differential equations. In this paper I review the use of numerical continuation for studying these solutions. I discuss the calculation of the form and stability of a given periodic travelling wave, and the calculation of boundaries in a two-dimensional parameter plane for wave existence and stability. I also describe the automated implementation of these numerical continuation procedures via the software package WAVETRAIN (http://www.ma.hw.ac.uk/wavetrain). I conclude by discussing ongoing work on numerical continuation methods for determining the absolute stability of periodic travelling waves.

1. Introduction

Periodic travelling waves (wavetrains) are an important solution type for many partial differential equations (PDEs). They play a fundamental role in the one-dimensional behaviour of self-oscillatory systems [1], for which the complex Ginzburg–Landau equation is the prototype example [2], and they arise in a wide range of other equations including excitable systems [3] and reaction–diffusion–advection equations [4]. As well as this fundamental mathematical role, periodic travelling waves (PTWs) occur in many applications. In physics, PTWs play an important role in hydrodynamics [5–7] and solar cycles [8]. In chemistry, travelling bands were observed in the Belousov–Zhabotinskii reaction more than 30 years ago [9] and are part of the wide range of behaviours seen in oscillatory and excitable chemical reactions [10–12]. In ecology, PTWs have been identified in spatiotemporal data sets on a number of cyclic populations [13–15], and occur on a landscape scale in semi-arid environments, where bands of vegetation moving slowly uphill on gentle slopes are a characteristic feature [16,17].

Like all travelling wave solutions, PTWs are functions of the single variable $z = x - ct$; here t and x are the time and (one-dimensional) space coordinates, and c is the wave speed. This solution ansatz reduces the PDEs to an ordinary differential equation (ODE) system, and a PTW is a limit cycle solution of these ODEs. However the limit cycle is typically unstable as an ODE solution in both the positive and negative z directions, meaning that it cannot be calculated by direct numerical integration of the ODEs. The standard method for calculating a PTW solution is therefore numerical continuation: starting from a Hopf bifurcation in the travelling wave equations, one follows the limit cycle branch until the required PTW solution is reached. Extensions of this approach enable calculation of the regions of parameter space in which PTWs exist. Stability of a PTW can also be determined via numerical continuation, using the method of Rademacher et al. [18] for computing the essential spectrum. In this paper, I will review these different applications of numerical continuation to the study of PTWs, illustrating my remarks via the calculation of boundaries in parameter space for the existence and stability of PTWs in the Klausmeier model for banded vegetation in semi-arid environments [19].

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Some of the methods that I will describe are relatively difficult to implement, even when using an established numerical continuation program such as AUTO [20–23], and this has been a significant barrier to their wider use. In Section 4, I will describe a new software package called WAVETRAIN which uses AUTO to study PTW solutions in an easy-to-use automated way.

2. The Klausmeier model for banded vegetation

In semi-arid environments, vegetation is often self-organised into spatial patterns. A particularly striking manifestation of this is vegetation banding on gentle slopes, in which stripes of grass, shrubs or trees run parallel to the contours, alternating with regions of bare ground [16,17]. A number of mathematical models have been developed for banded vegetation, reflecting the various ecological mechanisms that have been proposed. I will illustrate my discussion of the study of PTWs via the Klausmeier model [19], which has the dimensionless form

$$\partial p/\partial t = \overbrace{\omega p^2}^{\text{plant growth}} - \overbrace{Bp}^{\text{plant loss}} + \overbrace{\partial^2 p/\partial x^2}^{\text{plant dispersal}}, \tag{1a}$$

$$\partial \omega/\partial t = \underbrace{A}_{\text{rain-fall}} - \underbrace{\omega}_{\text{evaporation}} - \underbrace{\omega p^2}_{\text{uptake by plants}} + \underbrace{v\partial\omega/\partial x}_{\text{flow downhill}}. \tag{1b}$$

This was the first PDE model to be proposed for banded vegetation; p and ω are the densities of plant biomass and water, respectively. The parameters A, B and v are dimensionless combinations of a number of ecological parameters, but can be most conveniently interpreted as reflecting mean annual rainfall, plant loss including herbivory, and slope gradient, respectively. A key component of (1) is the nonlinear term ωp^2, which reflects the fact that higher levels of organic matter in the soil, and the presence of roots, increase the infiltration of rain water into the soil [24,25]. A detailed ecological appraisal of the Klausmeier model is given in [26], and mathematical properties of the equations are discussed in [27–30]. Before proceeding, it is important to emphasise that (1) is only one of a number of different mathematical models for banded vegetation; Refs. [31–35] contain a selection of other models, and [36] reviews the modelling literature in this area.

Banded vegetation corresponds to spatial patterns of (1) that move in the positive x direction (uphill) at a constant speed, that is, PTWS. Therefore it is important to understand the constraints on the parameters (A, B, v and wave speed) for PTW solutions of (1) to exist, and for them to be stable.

3. Application of numerical continuation to periodic travelling waves

In this section, I will describe the basic uses of numerical continuation to study the form and existence (Section 3.1) and stability (Section 3.2) of PTW solutions of the partial differential equations

$$\partial \underline{u}/\partial t = \underline{F}(\underline{u}, \partial\underline{u}/\partial x, \partial^2\underline{u}/\partial x^2, \ldots). \tag{2}$$

I will illustrate my remarks via the Klausmeier model (1); the results I will present on existence of PTWS for (1) have been shown previously in [4], but those on wave stability are new. Throughout this section I will consider only typical, simple behaviour, for which the Klausmeier model provides a good example for the parameter values considered. Moreover, to avoid interfering with readability I will for the most part omit caveats about the possibility of more complicated cases. Rather, in Section 5, I will describe various complications that can arise, and how they can be overcome.

3.1. Periodic travelling wave form and existence

Travelling wave solutions of (2) satisfy

$$c\,d\underline{U}/dz + \underline{F}\left(\underline{U}, d\underline{U}/dz, d^2\underline{U}/dz^2, \ldots\right) = 0 \tag{3}$$

where $\underline{u}(x,t) = \underline{U}(z)$ with $z = x - ct$; c is the wave speed. As discussed in Section 1, a PTW is a limit cycle solution of (3). In simple cases, the limit cycle branch containing this solution is monotonic in the parameters, and has at least one end terminating at a Hopf bifurcation point. For example, Fig. 1 illustrates the branch of PTW solutions of (1) with speed $c = 2$, as a function of the rainfall parameter A. The solution branch emanates from a Hopf bifurcation point at $A \approx 2.78$, and terminates at a homoclinic solution at $A \approx 0.32$. To calculate a PTW solution for a value of A between these limits, one therefore begins by performing a numerical continuation of the steady state, looking for the Hopf bifurcation point. One then switches to the limit cycle solution branch, and numerically continues this branch until the required value of A is reached.

PTW solutions depend on the parameters in the original PDES (2) and also on the wave speed c. If PTW solutions exist for a given set of PDE parameters, then they will do so for a range of values of c [1], with the value of c relevant to a particular PDE solution depending on initial and boundary conditions. Therefore it is natural to consider PTW existence in a parameter plane whose axes are the wave speed c and one of the PDE parameters, referred to henceforth as the "control parameter". For examples of PTW existence visualised in this way, see Refs. [4,39,40].