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On uniqueness and decay of solution for Hirota equation

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ABSTRACT

We address the question of the uniqueness of solution to the initial value problem associated to the equation

$$\partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + i\gamma|u|^2 u + \delta|u|^2\partial_x u + \epsilon u^2\partial_x\bar{u} = 0, \quad x,\ t \in \mathbb{R},$$

and prove that a certain decay property of the difference $u_1 - u_2$ of two solutions u_1 and u_2 at two different instants of times $t = 0$ and $t = 1$, is sufficient to ensure that $u_1 = u_2$ for all the time.

1. Introduction

In this work we consider the following equation:

$$\partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + i\gamma|u|^2 u + \delta|u|^2\partial_x u + \epsilon u^2\partial_x\bar{u} = 0, \quad x, t \in \mathbb{R}, \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}, \beta \neq 0, \gamma, \delta, \epsilon \in \mathbb{C}$ and $u = u(x,t)$ is a complex valued function. Our main concern is to find a sufficient decay property satisfied by the difference of two different solutions at two different instants of time to prove the uniqueness of the solution to the initial value problem (IVP) associated to (1.1).

The Eq. (1.1), with the mixed structure of Korteweg–de Vries (KdV) and Schrödinger equations, was proposed by Hasegawa and Kodama [7,16] to describe the nonlinear propagation of pulses in optical fibers. This equation is also known as Hirota equation in the literature. Several aspects of this equation including well-posedness issues, solitary wave solutions, unique continuation property, have been studied by various authors recently, see for example [2–4,17,22], and references therein.

Study of unique continuation property (UCP) for certain models has drawn much attention of a considerable section of mathematicians in recent time, see for example [1,4,8–15,18–21,23,24] and references therein. In particular, in [3,4] we addressed the UCP for the Eq. (1.1). In [4], we proved that if a sufficiently smooth solution u to the initial value problem associated to (1.1) is supported in a half line at two different instants of time then u vanishes identically. The precise statement of our result in [4] is the following.

Theorem 1.1 [4]. *Let $u \in C([t_1, t_2]; H^s) \cap C^1([t_1, t_2]; H^1)$, $s \geqslant 4$ be a solution of the Eq. (1.1) with $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R},\ \beta \neq 0$. If there exists $t_1 < t_2$ such that*

$$\operatorname{supp} u(\cdot, t_j) \subset (-\infty, a), \quad j = 1, 2, \tag{1.2}$$

$$or, \ (\operatorname{supp} u(\cdot, t_j) \subset (b, \infty), \quad j = 1, 2). \tag{1.3}$$

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Then $u(t) = 0$ for all $t \in [t_1, t_2]$.

In our subsequent work [3], we obtained more general uniqueness property for the solution of the IVP associated to (1.1).

Theorem 1.2 [3]. *Let $u, v \in C([t_1,t_2];H^s) \cap C^1([t_1,t_2];H^1)$, $s \geqslant 4$ be solutions of the* Eq. (1.1) *with $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$, $\beta \neq 0$. If there exists $b \in \mathbb{R}$ such that*

$$u(x,t) = v(x,t), \quad (x,t) \in (b,\infty) \times \{t_1,t_2\}, \tag{1.4}$$

$$or, (u(x,t) = v(x,t), \quad (x,t) \in (-\infty,b) \times \{t_1,t_2\}). \tag{1.5}$$

Then

$$u(t) = v(t) \quad \forall t \in [t_1, t_2].$$

Remark 1.1. Theorem 1.1 is the special case of Theorem 1.2 when $v \equiv 0$.

Motivation to obtain the above results is the following observation. Consider the IVP associated to the linear part of (1.1), i.e.,

$$\begin{cases} u_t + i\alpha u_{xx} + \beta u_{xxx} = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.6}$$

If u and v are solutions to (1.6) then $w := u - v$ is also a solution to (1.6) with initial data $w(x,0) = u(x,0) - v(x,0) =: w_0(x)$. If w_0 is sufficiently smooth and has compact support, then using Paley–Wiener theorem it is easy to see (for detail see [4]) that $w \equiv 0$, i.e., $u \equiv v$. But the proof of the same property is not so simple when one considers the nonlinear terms as well, because in this case $w := u - v$ is no more a solution. To overcome this situation, we generalized and employed the techniques developed in the context of the generalized KdV equation by Kenig–Ponce–Vega [12,13] to prove Theorems 1.1 and 1.2.

Quite recently, Escauriaza et al. [6] introduced a new technique to obtain sufficient conditions on the behavior of the difference $u_1 - u_2$ of two solutions u_1 and u_2 of the generalized KdV equation at two different instants of time $t = 0$ and $t = 1$ that guarantees $u_1 \equiv u_2$. In [6], the authors obtained a sharp decay condition to guarantee the uniqueness of solution to the generalized KdV equation. So, there arise a natural question, whether one can find such a decay condition to get uniqueness property for a mixed equation of the KdV and Schrödinger type. In this work, we shall extend the approach in [6] to address a uniqueness question to the IVP associated to the Hirota Eq. (1.1) which has a mixed structure of the KdV and the Schrödinger equations. Our first main result of this work is the following.

Theorem 1.3. *Let $u_1, u_2 \in C([0,1]; H^3(\mathbb{R})) \cap L^2(|x|^2dx))$, be strong solutions of the Eq.* (1.1) *with $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$, $\beta \neq 0$. If, for any $a > 0$,*

$$u_1(\cdot,0) - u_2(\cdot,0), \quad u_1(\cdot,1) - u_2(\cdot,1) \in H^1(e^{ax_+^{3/2}}dx), \tag{1.7}$$

then

$$u_1 \equiv u_2.$$

To prove Theorem 1.3 we follow the techniques introduced in [6] by deriving some new estimates that are appropriate to work with the equation under consideration.

Using the gauge transformation

$$v(x,t) = e^{i\lambda x + i(\alpha\lambda^2 - 2\beta\lambda^3)t} u(2\alpha\lambda - 3\beta\lambda^2)t, t) \tag{1.8}$$

with $\lambda = \frac{\alpha}{3\beta}$, one can work for an equivalent equation for v without the term $i\alpha u_{xx}$ in the linear part. With this, it seems that, our result for the original equation also follows from the techniques in [6]. But it is not the case: because if we work on the transformed equation for v (without the Schrödinger term), in the beginning we need to suppose that

$$v_1(x,t_j) - v_2(x,t_j) \in H^1(e^{ax_+^{3/2}}) =: X_1, \quad j = 1,2. \tag{1.9}$$

So, after undoing the transformation, for the original solution u, we need:

$$u_1(x,t_j) - u_2(x,t_j) \in H^1(e^{ax_+^{3/2}}) \iff v_1(x,t_j) - v_2(x,t_j) \in H^1\left(e^{a(x+\alpha^2/2\beta)_+^{3/2}}\right) =: X_2. \tag{1.10}$$

But (1.10) is not always true, because one can find function f for which $\|f\|_{X_1} < \infty$ but $\|f\|_{X_2} = \infty$ and vice versa.

Therefore, it is not possible to discard the Schrödinger term using gauge transform so as to apply the techniques from [6] directly in our case.

On the other hand, one may think of treating the term $i\alpha u_{xx}$ at par with nonlinear terms and apply the estimates from [6] directly. This is also not possible, because the term with α does not satisfy the necessary decay condition so as to use the estimates from our earlier works [3,4]. This situation has been explained in our earlier work [3], Remark 3.10.

Although the idea and estimates are similar to the one introduced in [6], the presence of the Schrödinger term in the linear part creates obstacle to obtain such estimates, which can be seen more explicitly in the derivation of the lower estimates