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On finite difference methods for fourth-order fractional diffusion–wave and subdiffusion systems ☆

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ABSTRACT

In this paper, firstly, the finite difference method is explored for the fourth-order fractional diffusion–wave system. The method is proved to be uniquely solvable, stable and convergent in l_∞-norm by the energy method. Then we examine a subdiffusion system and present the numerical analysis using a different method. Numerical experiments are provided to demonstrate the accuracy and efficiency of the proposed schemes.

1. Introduction

Fractional diffusion equations (FDEs) have been studied widely in recent years. Investigations show that FDEs can describe many phenomena and processes in physics [1,2], engineering [3,4], and other sciences [5–8].

The fractional diffusion–wave (FDW) equation is a linear integro-differential equation obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative by a fractional derivative of order $\alpha > 0$ [9,10]. Much considerable work has been done theoretically or numerically on the fractional diffusion–wave or the subdiffusion equations, including the finite difference methods [11–20]. Recently, Tadjern and Meerschaert [21,22] firstly increased the temporal accuracy to second-order by using an extrapolated Crank–Nicolson method for the one-dimensional fractional diffusion equation and a Richardson extrapolation method for the two-dimensional fractional diffusion equation. Cui [23] constructed a compact finite difference scheme with spatial accuracy of fourth order for the one-dimensional fractional diffusion equation. Liu et al. [24] established a new implicit difference scheme for a modified anomalous subdiffusion equation with a nonlinear source term and analyzed the stability and convergence by using a new energy method. Shen et al. [25] studied the fractional Fokker–Planck equation and presented some practical numerical methods. Sun and Wu [26] investigated a fully discrete scheme by the method of order reduction for a FDW system and proved the solvability, stability and convergence by the energy method. Du et al. [27] provided a compact difference scheme for the fractional diffusion–wave system on the basis of [26]. Gao and Sun [28] derived a compact finite difference scheme for the fractional subdiffusion equations with convergence order $O(\tau^{2-\gamma} + h^4)$. Zhao and Sun [29] presented a box-type scheme by using order reduction approach and L1 discretization and applied a novel technique in the proof of both stability and convergence.

During the research on modeling and controlling of beams and waves, to reduce the level of noise, researchers have been led to consider control laws based on lower order derivatives, that is, fractional derivatives [30]. Then another strong motivation for considering fractional derivatives is that, for a Bemouilli–Euler beam, the H_∞ optimal wave absorbing problem at

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the boundary is solved by a fractional derivative compensator of order one-half [30]. But in some applications, a fourth-order space derivative term must be indispensable. For example, wave propagation in beams and modeling formation of grooves on a flat surface because of grain require fourth-order space derivative terms in their formulations [31,32]. Jafari et al. [33] solved a fourth-order FDW equation in a bounded domain by decomposition method. Agarwal [34,35] presented a general solution to FDE equations containing fourth-order space derivative in unbounded and bounded domains. Golbabai and Sayevand [36] applied homotopy perturbation method to obtain the approximate solution of the generalized fourth-order fractional diffusion–wave equations. So far, the finite difference method for the fourth-order fractional diffusion systems is few to be seen. Here, we explore this method for the fourth-order fractional diffusion systems.

In this paper, firstly, we consider the fourth-order fractional diffusion–wave equation [35]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + b^2 \frac{\partial^4 u}{\partial x^4} = f(x,t), \quad x \in (0,L),\ t \in (0,T] \tag{1}$$

with the boundary conditions

$$u(0,t) = g_1(t), \quad u(L,t) = g_2(t), \quad \frac{\partial^2 u(0,t)}{\partial x^2} = h_1(t), \quad \frac{\partial^2 u(L,t)}{\partial x^2} = h_2(t), \quad t \in (0,T] \tag{2}$$

and the initial conditions

$$u(x,0) = \phi(x), \quad \frac{\partial u(x,0)}{\partial t} = \psi(x), \quad x \in [0,L], \tag{3}$$

where α is a parameter describing the order of the fractional derivative, b denotes a constant coefficient and $f(x,t)$ is a known function. As in [35], we here consider Caputo fractional derivative

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(x,s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}}, \quad 1 < \alpha < 2 \tag{4}$$

with Γ denoting the gamma function. When $\alpha = 1$ and $\alpha = 2$, Eq. (1) represents a diffusion and a wave equation respectively. It is observed that as α increases from 0 to 2, the process changes from (slow) subdiffusion ($0 < \alpha < 1$) to classical diffusion ($\alpha = 1$) to diffusion–wave ($1 < \alpha < 2$) to classical wave process ($\alpha = 2$) [32].

The rest of the paper is organized as follows. In Section 2, one Crank–Nicolson finite difference scheme is derived. Section 3 is devoted to prove the solvability, stability, and convergence of the scheme by using the energy method. In Section 4, a subdiffusion system is examined and a finite difference scheme as well as the numerical analysis is provided. Section 5 demonstrates three illustrative numerical examples. In the end, a concise conclusion is made.

2. Finite difference scheme for diffusion–wave system ($1 < \alpha < 2$)

Firstly, we denote $x_j = jh$, $t_n = n\tau$, $\Omega_h = \{x_j | 0 \leqslant j \leqslant M\}$, $\Omega_\tau = \{t_n | 0 \leqslant n \leqslant N\}$, and $\Omega_h^\tau = \Omega_h \times \Omega_\tau$, where $h = L/M$, $\tau = T/N$ are the uniform spacial and temporal mesh sizes respectively, and M, N are two positive integers.

Suppose $u = \{u_j^n | 0 \leqslant j \leqslant M,\ 0 \leqslant n \leqslant N\}$ is a grid function on Ω_h^τ. Introduce the following notations:

$$u_j^{n-\frac{1}{2}} = \frac{1}{2}\left(u_j^n + u_j^{n-1}\right), \quad \delta_t u_j^{n-\frac{1}{2}} = \frac{1}{\tau}\left(u_j^n - u_j^{n-1}\right),$$
$$\delta_x u_{j-\frac{1}{2}}^n = \frac{1}{h}\left(u_j^n - u_{j-1}^n\right), \quad \delta_x^2 u_j^n = \frac{1}{h^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right),$$
$$\delta_x^4 u_j^n = \delta_x^2\left(\delta_x^2 u_j^n\right) = \frac{1}{h^4}\left(u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n\right).$$

Let $u_h = \{u = (u_0, u_1, \ldots, u_M), u_0 = u_M = 0\}$. For $v \in u_h$, $u \in u_h$, we introduce the following discrete l^2-inner product$(\cdot,\cdot)$, discrete l^2-norm $\|\cdot\|$ and discrete l_∞-norm $\|\cdot\|_\infty$:

$$(u^n, u^n) = h\sum_{j=1}^{M-1} u_j^n u_j^n, \quad \|u^n\|^2 = (u^n, u^n), \quad \|u^n\|_\infty = \max_{1 \leqslant j \leqslant M-1} |u_j^n|,$$
$$\|\delta_x u^n\|^2 = h\sum_{j=1}^{M}\left(\delta_x u_{j-\frac{1}{2}}^n\right)^2, \quad \|\delta_x^2 u^n\|^2 = h\sum_{j=1}^{M-1}\left(\delta_x^2 u_j^n\right)^2.$$

In the paper, C denotes a general positive constant, which may have different values in different occurrences.

Now, we consider the finite difference scheme for the problem (1)–(3). Assume that problem (1)–(3) has a smooth solution $u(x,t) \in C_{x,t}^{4,3}([0,L] \times (0,T])$. Let

$$v(x,t) = \frac{\partial u(x,t)}{\partial t}, \tag{5}$$