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Reliability and non-reliability studies of Poisson variables in series and parallel systems

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ABSTRACT

In this paper, we have derived the distribution of the minimum and maximum of two independent Poisson random variables. A useful procedure for computing the probabilities is given and a total of four numerical examples are presented. Of these four examples, the first two are on the generated data and the other two are on the Champion League Soccer data in order to illustrate the model which is considered here. The hazard rate and the reversed hazard rate, of the minimum and maximum of two independent discrete random variables, are also obtained and their monotonicity is investigated. The results for the Poisson-distributed variables are obtained as special cases.

1. Introduction and motivation

Samaniego [8] studied the distribution of a convoluted Poisson random variable given by $X = Y + Z$, where Y has a Poisson distribution and Z is a nonnegative integer-valued random variable independent of Y. Examples of its use include regression problems with both variables subject to error and *signal plus noise* models in statistical communication theory. In this paper, we consider two independent Poisson variables from the reliability theory viewpoint. For example, in weapons reliability, X_1 and X_2 represent the number of rounds fired until failure and we observe

$$T_1 := \min(X_1, X_2) \qquad \text{or} \qquad T_2 := \max(X_1, X_2).$$

For other applications of discrete models in reliability theory, see the earlier investigations by (for example) Shaked et al. [9,10] and Gupta et al. [2,3].

Motivated essentially by the above-mentioned investigations, our main interest in this paper is to obtain the exact distribution of T_1 and T_2 and to examine the monotonicity of the failure rates of T_1 and T_2, assuming that X_1 and X_2 are independent Poisson variables with means λ_1 and λ_2, respectively. Thus, in Section 2, we obtain the distribution of T_1 and T_2 when X_1 and X_2 are independent Poisson variables. The probability generating functions of T_1 and T_2 are also provided. Section 3 contains a useful procedure for computing the probabilities. In Section 4, this procedure is employed to present four illustrative examples, two on the generated data and two on the Champion League Soccer data. Section 5 contains some general results for the hazard rate and the reversed hazard rate of two independent discrete variables. The monotonicity of the hazard rates is investigated and the results for independent Poisson variables are also established.

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2. Distributions of T_1 and T_2

Before proceeding to present a potentially useful result as the following Lemma, we recall the definition of the modified Bessel function $I_\nu(z)$ of the first kind as follows (see, for example, [12, p. 28 *et seq.*]):

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{j=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^j}{j!\Gamma(\nu+j+1)} = \frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\text{—};\nu+1;\frac{1}{4}z^2\right) \tag{1}$$

in terms of the gamma function $\Gamma(z)$ and the familiar confluent hypergeometric function ${}_0F_1$ (see, for details, [12, p. 23, Section II.3]).

Lemma. *Let X_1 and X_2 be two independent Poisson-distributed random variables with means λ_1 and λ_2, respectively. Then*

$$P(X_1 - X_2 = k) = e^{-(\lambda_1+\lambda_2)}\left(\frac{\lambda_1}{\lambda_2}\right)^{k/2} \quad I_k\left(2\sqrt{\lambda_1\lambda_2}\right), \tag{2}$$

$$(-\infty < k < \infty;\ k \in \mathbb{Z}),$$

where $I_\nu(z)$ is the modified Bessel function of the first kind of order ν given by (1), $\mathbb{Z}$ being the set of integers.

Proof. For the proof of the Lemma, we refer to the recent work by Ong et al. [6]. □

Remark 1. The above result was obtained much earlier by several authors including Skellam [11], DeCastro [1] and Prekopa [7].

In what follows, we present the distributions of T_1 and T_2 given (as before) by

$$T_1 := \min(X_1, X_2) \qquad \text{and} \qquad T_2 := \max(X_1, X_2).$$

Theorem 1. *Let X_1 and X_2 be two independent Poisson-distributed random variables with means λ_1 and λ_2. Then*

$$P(T_1 = t) = \frac{e^{-(\lambda_1+\lambda_2)}}{t!}\left[e^{-\lambda_1}\lambda_1^t \sum_{k=-\infty}^{-1}\left(\frac{\lambda_1}{\lambda_2}\right)^{k/2} I_k\left(2\sqrt{\lambda_1\lambda_2}\right) + e^{-\lambda_2}\lambda_2^t \sum_{k=-\infty}^{-1}\left(\frac{\lambda_2}{\lambda_1}\right)^{k/2} I_k\left(2\sqrt{\lambda_1\lambda_2}\right) + \frac{(\lambda_1\lambda_2)^t}{t!}\right]. \tag{3}$$

Proof. It is easily seen that

$$P(T_1 = t) = P(X_1 = t)P(X_1 < X_2) + P(X_2 = t)P(X_1 > X_2) + P(X_1 = t)P(X_2 = t) = P(X_1 = t)P(X_1 - X_2 < 0) + P(X_2 = t)P(X_1 - X_2 > 0) + P(X_1 = t)P(X_2 = t).$$

The result (3) can thus be obtained by applying the results asserted by the above Lemma. □

Theorem 2. *Let X_1 and X_2 be two independent Poisson-distributed random variables with means λ_1 and λ_2, respectively. Then*

$$P(T_2 = t) = \frac{e^{-(\lambda_1+\lambda_2)}}{t!}\left[e^{-\lambda_1}\lambda_1^t \sum_{k=-\infty}^{-1}\left(\frac{\lambda_2}{\lambda_1}\right)^{k/2} I_k\left(2\sqrt{\lambda_1\lambda_2}\right) + e^{-\lambda_2}\lambda_2^t \sum_{k=-\infty}^{-1}\left(\frac{\lambda_1}{\lambda_2}\right)^{k/2} I_k\left(2\sqrt{\lambda_1\lambda_2}\right) + \frac{(\lambda_1\lambda_2)^t}{t!}\right]. \tag{4}$$

Proof. It is not difficult to observe that

$$P(T_2 = t) = P(X_1 = t)P(X_2 < X_1) + P(X_2 = t)P(X_1 < X_2) + P(X_1 = t)P(X_2 = t) = P(X_1 = t)P(X_2 - X_1 < 0) + P(X_2 = t)P(X_1 - X_2 < 0) + P(X_1 = t)P(X_2 = t),$$

which yields the result (4) by applying the above Lemma. □

2.1. Probability generating function of T_1

The probability generating function of T_1 is given by