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The new upper bounds on the spectral radius of weighted graphs

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ABSTRACT

Let us consider weighted graphs, where the weights of the edges are positive definite matrices. The eigenvalues of a weighted graph are the eigenvalues of its adjacency matrix and the spectral radius of a weighted graph is also the spectral radius of its adjacency matrix. In this paper, we obtain two upper bounds for the spectral radius of weighted graphs and compare with a known upper bound. We also characterize graphs for which the upper bounds are attained.

1. Introduction

We consider simple graphs, that is, graph which have no loops or parallel edges. Hence a graph $G = (V,E)$ consists of a finite set of vertices, V, and a set of edges, E, each of whose elements is an unordered pair of distinct vertices. Generally V is taken as $V = \{1,2,\ldots,n\}$.

A weighted graph is a graph, each edge of which has been assigned a number. Such weights might represent, for example, costs, lengths or capacities, etc. The weight of the edge can also be a square matrix. In this paper, unless otherwise stated, the weights of the edges will be taken positive definite matrices of the same order.

Now we introduce some notations. Let G be a weighted graph on n vertices, denote by $w_{i,j}$ the positive definite matrix of order p of the edge ij, and assume that $w_{i,j} = w_{j,i}$. We write $i \sim j$ if vertices i and j are adjacent. Let $w_i = \sum_{j:j\sim i} w_{i,j}$ be the weight matrix of the vertex i.

The adjacency matrix of a graph G is a block matrix, denoted and defined as $A(G) = (a_{ij})$ where

$$a_{i,j} = \begin{cases} w_{i,j} & \text{if } i \sim j, \\ 0 & \text{otherwise}. \end{cases}$$

Note that in the definition above, the zero denotes the $p \times p$ zero matrix. Thus $A(G)$ is a square matrix of order np. For any symmetric matrix K, let $\rho_1(K)$ denote the largest eigenvalue, in modulus (i.e., the spectral radius) of K.

Let $G = (V,E)$ be, if V is the disjoint union of two nonempty sets V_1 and V_2 such that every vertex i in V_1 has the same $\rho_1(w_i)$ and every vertex j in V_2 has the same $\rho_1(w_j)$, then G will be called a weight-semiregular graph. If $\rho_1(w_i) = \rho_1(w_j)$ in weight-semiregular graph, then G will be called a weight-regular graph.

Upper and lower bounds for the spectral radius of unweighted graphs have been investigated to a great extent in literature [1–5,7,8]. Especially, some of the authors [4,5] have discussed whether the bounds, which they have obtained, are sharper bounds for the spectral radius of graphs. In addition to the studies about unweighted graphs, some studies on weighted graphs have also been done. Das and Bapat [6] have studied weighted graphs, where the weights of the edges are positive definite matrices, and found an upper bound for the spectral radius of weighted graphs. They have also characterized graphs for which

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the upper bound is attained. The main result of this paper, contained in Section 2, gives two upper bounds on the spectral radius for weighted graphs, where the edge weights are positive definite matrices. We also compare our upper bounds with Das and Bapat's upper bound. We call our upper bounds as new, because they are sharper than Das and Bapat's upper bound.

2. The new upper bounds on the spectral radius of weighted graphs

Lemma 1 [9]. *Let B be a Hermitian $n\times n$ matrix with ρ_1 as its largest eigenvalue, in modulus. then for any $\bar{x}\in R^n(\bar{x}\neq\bar{0}),\ \bar{y}\in R^n(\bar{y}\neq\bar{0})$, the spectral radius $|\rho_1|$ satisfies*

$$|\bar{x}^T B\bar{y}| \leqslant |\rho_1|\sqrt{\bar{x}^T\bar{x}}\sqrt{\bar{y}^T\bar{y}} \tag{2.1}$$

Equality holds if and only if $\bar{x}$ is an eigenvector of B corresponding to ρ_1 and $\bar{y}=\alpha\bar{x}$ for some $\alpha\in R$.

Lemma 2 ([9]). *Let A, $B\in M_n$ be Hermitian and let the eigenvalues $\rho_i(A)$, $\rho_i(B)$, and $\rho_i(A+B)$ be arranged in increasing order ($\rho_n\leqslant\rho_{n-1}\leqslant\cdots\leqslant\rho_2\leqslant\rho_1$). For each $k=1,2,\ldots,n$ we have*

$$\rho_k(A)+\rho_n(B)\leqslant\rho_k(A+B)\leqslant\rho_k(A)+\rho_1(B) \tag{2.2}$$

Lemma 3 ([6]). *Let $B_1, B_2,\ldots,B_k$ be positive definite matrices of order n and let $B=\sum_{i=1}^n B_i$. If $\bar{x}$ is an eigenvector of each B_i corresponding to the largest eigenvalue $\rho_1(B_i)$ for all i, then $\bar{x}$ is also an eigenvector of B corresponding to the largest eigenvalue $\rho_1(B)$.*

Theorem 1 ([6]). *Let G be a weighted graph which is simple, connected and let ρ_1 be the largest eigenvalue (in modulus) of G, so that $|\rho_1|$ is the spectral radius of G. Then*

$$|\rho_1|\leqslant\max_{i\sim j}\left\{\sqrt{\sum_{k:k\sim i}\rho_1(w_{i,k})\sum_{k:k\sim j}\rho_1(w_{j,k})}\right\}, \tag{2.3}$$

where $w_{i,j}$ is the positive definite matrix of order p of the edge ij. Moreover, equality holds in (2.3) if and only if

(i) *G is a weighted-regular graph or G is a weight-semiregular bipartite graph;*
(ii) *$w_{i,j}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{i,j})$ for all i, j.*

Theorem 2. *Let G be a weighted graph which is simple, connected and let ρ_1 be the largest eigenvalue (in modulus) of G, so that $|\rho_1|$ is the spectral radius of G. Then*

$$|\rho_1|\leqslant\max_{i\sim j}\left\{\sqrt{\sum_{k:k\sim i}\frac{\rho_1(w_k)}{\rho_1(w_i)}\rho_1(w_{i,k})\sum_{k:k\sim j}\frac{\rho_1(w_k)}{\rho_1(w_j)}\rho_1(w_{j,k})}\right\}, \tag{2.4}$$

where $w_{i,j}$ is the positive definite matrix of order p of the edge ij. Moreover equality holds in (2.4) if and only if

(i) *G is a weighted-regular graph or G is a weight-semiregular bipartite graph;*
(ii) *$w_{i,j}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{i,j})$ for all i, j.*

Proof. Let $M(G)$ be the block diagonal matrix $diag(\gamma_1 I_{p,p}, \gamma_2 I_{p,p},\ldots,\gamma_n I_{p,p})$ where $\gamma_i=\rho_1(w_i)$, $i=1,2,\ldots,n$.

Let $\overline{X}=(\bar{x}_1^T,\bar{x}_2^T,\ldots,\bar{x}_n^T)^T$ be an eigenvector corresponding to the largest eigenvalue ρ_1 of $M(G)^{-1}A(G)M(G)$. We assume that $\bar{x}_i$ is the vector component of $\overline{X}$ such that $\bar{x}_i^T\bar{x}_i=\max_{k\in V}\{\bar{x}_k^T\bar{x}_k\}$. Since $\overline{X}$ is nonzero, so is $\bar{x}_i$.

Let $\bar{x}_j^T\bar{x}_j=\max_{k:k\sim i}\{\bar{x}_k^T\bar{x}_k\}$ be, then, for all k, $i\sim k$, we get

$$\bar{x}_j^T\bar{x}_j\geqslant\bar{x}_k^T\bar{x}_k. \tag{2.5}$$

The (i,j)th block of $M(G)^{-1}A(G)M(G)$ is

$$\begin{cases}\frac{\gamma_j}{\gamma_i}w_{i,j} & \text{if } i\sim j,\\ 0 & \text{otherwise}.\end{cases}$$

We have

$$\{M(G)^{-1}A(G)M(G)\}\overline{X}=\rho_1\overline{X}. \tag{2.6}$$

From the ith equation of (2.6), we have

$$\rho_1\bar{x}_i=\sum_{k:k\sim i}\frac{\gamma_k}{\gamma_i}w_{i,k}\bar{x}_k, \tag{2.7}$$

i.e., $\rho_1\bar{x}_i\bar{x}_i^T=\sum_{k:k\sim i}\bar{x}_i^T\frac{\gamma_k}{\gamma_i}w_{i,k}\bar{x}_k$