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Asymptotic convergence of the solutions of a discrete equation with several delays

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ABSTRACT

A discrete equation

$$\Delta y(n) = \sum_{i=1}^{s} \beta_i(n)[y(n-j_i) - y(n-k_i)]$$

with integer delays k_i and j_i, $k_i > j_i \geqslant 0$ is considered. It is assumed that $\beta_i : \mathbb{Z}_{n_0-k}^{\infty} \to [0,\infty), n_0 \in \mathbb{Z}, k = \max\{k_1, k_2, \ldots, k_s\}, \mathbb{Z}_p^{\infty} := \{p, p+1, \ldots\}, p \in \mathbb{Z}, \sum_{i=1}^{s} \beta_i(n) > 0, s \in \mathbb{Z}_1^{\infty}$ is a fixed integer and $n \in \mathbb{Z}_{n_0}^{\infty}$. Criteria of asymptotic convergence of solutions are expressed in terms of inequalities for the functions β_i, $i = 1, \ldots, s$. Some general properties of solutions are derived as well. It is, e.g., proved that, for the asymptotical convergence of all solutions, the existence of a strictly monotone and asymptotically convergent solution is sufficient. A crucial role in the analysis of convergence is played by an auxiliary inequality derived from the form of a given discrete equation.

1. Introduction

For $p \leqslant q, p, q \in \mathbb{Z}$, we define the set $\mathbb{Z}_p^q := \{p, p+1, \ldots, q\}$. Throughout the paper we study asymptotic convergence of solutions of discrete equation with delays

$$\Delta y(n) = \sum_{i=1}^{s} \beta_i(n)[y(n-j_i) - y(n-k_i)] \tag{1}$$

as $n \to \infty$. In (1) we assume that integers k_i and j_i satisfy the inequality $k_i > j_i \geqslant 0, \beta_i : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}_+ := [0,\infty), i = 1, \ldots, s, n_0 \in \mathbb{Z}, k = \max\{k_1, k_2, \ldots, k_s\}, \sum_{i=1}^{s} \beta_i(n) > 0, n \in \mathbb{Z}_{n_0}^{\infty}, s \in \mathbb{Z}_1^{\infty}$ is a fixed integer and $\Delta y(n) := y(n+1) - y(n)$. Without loss of generality we assume $n_0 - k > 0$ (this is a technical detail necessary in some of the next inequalities).

Let $\mathcal{C} := \mathcal{C}(\mathbb{Z}_{-k}^{0}, \mathbb{R})$ be the Banach space of discrete functions mapping the discrete interval $\mathbb{Z}_{-k}^{0}$ into $\mathbb{R}$. Let $v \in \mathbb{Z}_{n_0}^{\infty}$ be fixed. The function $y : \mathbb{Z}_{v-k}^{\infty} \to \mathbb{R}$ is said to be a *solution of* (1) *on* $\mathbb{Z}_{v-k}^{\infty}$ if it satisfies Eq. (1) for every $n \in \mathbb{Z}_{v}^{\infty}$. A solution y of (1) on $\mathbb{Z}_{v-k}^{\infty}$ is *asymptotically convergent* if the limit $\lim_{n\to\infty} y(n)$ exists and is finite. We say that $y(v,\varphi)$, where $\varphi \in \mathcal{C}$, is a *solution of* (1) defined by the initial conditions (v,φ) if $y(v,\varphi)$ is a solution of (1) on $\mathbb{Z}_{v-k}^{\infty}$ and $y(v,\varphi)(v+m) = \varphi(m)$ for $m \in \mathbb{Z}_{-k}^{0}$.

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1.1. Problem of asymptotic convergence

We explain the problem of asymptotic convergence using a particular case of the Eq. (1) where the coefficients $\beta_i(n)$, $i=1,\dots,s$ are constant, i.e., we assume $\beta_i(n)=b_i=$ const, $i=1,\dots,s$, and consider an equation

$$\Delta y(n)=\sum_{i=1}^{s} b_i[y(n-j_i)-y(n-k_i)] \tag{2}$$

where $b_i\in\mathbb{R}_+$ and $\sum_{i=1}^{s} b_i>0$. We define an auxiliary positive constant

$$q := \sum_{i=1}^{s} b_i(k_i-j_i). \tag{3}$$

We show that its value plays an important role in the analysis of asymptotical convergence of solutions not only for Eq. (2), but for Eq. (1) as well. We will analyze the behavior of solutions of (3) in three important cases. Namely, we assume

(a) $q=1$,
(b) $q>1$,
(c) $q<1$.

1.1.1. The case $q = 1$

In accordance with the usual procedure, we are looking for a solution of (2) in the form $y(n)=\lambda^n, \lambda\in\mathbb{C}\setminus\{0\}$. After dividing by λ^n and multiplying by λ^k we get the characteristic equation

$$\lambda^{k+1}-\lambda^k=\sum_{i=1}^{s} b_i\left(\lambda^{k-j_i}-\lambda^{k-k_i}\right) \tag{4}$$

which obviously has a root $\lambda_1=1$. We show that this root has multiplicity 2. Dividing (4) by $(\lambda-1)$, we get

$$\lambda^k=\sum_{i=1}^{s} b_i\sum_{r=1}^{k_i-j_i}\lambda^{k-j_i-r} \tag{5}$$

For $\lambda=1$ its right-hand side equals

$$\sum_{i=1}^{s} b_i\sum_{r=1}^{k_i-j_i}\lambda^{k-j_i-r}\Bigg|_{\lambda=1}=\sum_{i=1}^{s} b_i\sum_{r=1}^{k_i-j_i}1=\sum_{i=1}^{s} b_i(k_i-j_i)=q=1.$$

Thus, $\lambda_2=1$. Then Eq. (2) has a family of divergent solutions $y(n)=Cn$ where C is an arbitrary constant, $C\neq 0$. We conclude that, in this case, not all solutions are asymptotically convergent.

1.1.2. The case $q > 1$

We rewrite Eq. (5) as

$$f(\lambda)=0 \tag{6}$$

where

$$f(\lambda) := \lambda^k-\sum_{i=1}^{s} b_i\sum_{r=1}^{k_i-j_i}\lambda^{k-j_i-r}$$

Since

$$f(1) := 1-\sum_{i=1}^{s} b_i\sum_{r=1}^{k_i-j_i}\lambda^{k-j_i-r}\Bigg|_{\lambda=1}=1-\sum_{i=1}^{s} b_i\sum_{r=1}^{k_i-j_i}1=1-\sum_{i=1}^{s} b_i(k_i-j_i)=1-q<0$$

and $f(+\infty)=+\infty$, we conclude that Eq. (6) and, consequently, the characteristic Eq. (4) has a root $\lambda>1$ and, consequently, a one-parametric family of divergent solutions. As in the previous case, we conclude that not all solutions are asymptotically convergent.

1.1.3. The case $q < 1$

We define two auxiliary functions

$$F(\lambda) := \lambda^k,\quad \Psi(\lambda) := -\sum_{i=1}^{s} b_i\sum_{r=1}^{k_i-j_i}\lambda^{k-j_i-r}$$