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Eigenvalue comparisons for a class of boundary value problems of discrete beam equation

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ABSTRACT

We study the structure of eigenvalues of a class of boundary value problems of fourth order difference equation

$$\Delta^4 y_i + b_{i+2}y_{i+2} = \lambda a_{i+2}y_{i+2}, \quad -1 \leqslant i \leqslant n-2,$$
$$\Delta y_0 = \Delta^3 y_{-1} = \Delta y_n = \Delta^3 y_{n-1} = 0.$$

The monotone behavior of the eigenvalues as the sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ change is obtained in this paper.

1. Introduction

Boundary value problems have important applications in physics, chemistry, and biology. For example, the boundary value problem

$$u^{(4)}(t) + p(t)u(t) = q(t)f(u(t)), \quad 0 \leqslant t \leqslant 1, \tag{1}$$
$$u'(0) = u'''(0) = u'(1) = u'''(1) = 0 \tag{2}$$

arises in the study of elasticity and has definite physical meanings. The Eq. (1) is often referred to as the beam equation, for it describes the deformation or deflection of an elastic beam under forces. The boundary conditions Eq. (2) mean that the beam is fastened with sliding clamps at both ends.

Consider a special case in which f is linear, say $f(u(t)) = \lambda u(t)$. In this case, we have the following boundary value problem for fourth order linear beam equation

$$y^{(4)}(t) + p(t)y(t) = \lambda q(t)y(t), \quad t \in [0,1], \quad y'(0) = y'''(0) = y'(1) = y'''(1) = 0. \tag{3}$$

For $n \geqslant 3$, define $h = 1/n$ and $t_i = ih$. For simplicity, define

$$y_i = y(t_i), \quad a_i = h^4 q(t_i), \quad b_i = h^4 p(t_i). \tag{4}$$

We discretize the differential equation in (3) at the node t_{i+2}, $-1 \leqslant i \leqslant n-2$. It is known that

$$y^{(4)}(t_{i+2}) \approx \frac{\Delta^4 y_i}{h^4},$$

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where the forward difference operator Δ is defined as $\Delta y_i = y_{i+1} - y_i$. Thus, the BVP ODE Eq. (3) can be discretized as

$$\left.\begin{array}{l}\Delta^4 y_i + b_{i+2}y_{i+2} = \lambda a_{i+2}y_{i+2}, \quad -1 \leqslant i \leqslant n-2,\\ \Delta y_0 = \Delta^3 y_{-1} = \Delta y_n = \Delta^3 y_{n-1} = 0,\end{array}\right\} \tag{5}$$

where λ is a parameter and a_i, b_i for $1 \leqslant i \leqslant n$ are defined in Eq. (4). Throughout this paper, we will study the boundary value problem of discrete beam Eq. (5) instead of its continuous counterpart Eq. (3). We assume that

(H) The $n \geqslant 3$ is a fixed integer. The $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ are finite sequences of real numbers such that $b_i \geqslant 0$ for $1 \leqslant i \leqslant n$ with $\sum_{i=1}^n b_i > 0$. And there is at least one non-zero member of the sequence $\{a_i\}_{i=1}^n$.

If λ is a number (maybe complex) such that Eq. (5) has a nontrivial solution $\{y_i\}_{i=-1}^{n+2}$, then λ is said to be an eigenvalue of the problem Eq. (5), and the corresponding nontrivial solution $\{y_i\}_{i=-1}^{n+2}$ is called an eigenvector of Eq. (5) corresponding to λ.

Some comparison results for the smallest eigenvalues of two eigenvalue problems for boundary value problems of $2n$th order linear differential equations were established in 1973 by Travis [10], by using the theory of u_0-positive linear operator in a Banach space equipped with a cone of "nonnegative" elements. Since then, some progress has been made on comparisons of eigenvalues of boundary value problems of differential equations and difference equations. In the early years, the focus was on the smallest eigenvalues only [2,3,10].

Recently, with the help of matrix theory, we managed to analyze the eigenvalue structure of some difference equations with boundary conditions and obtained comparison results for all eigenvalues [5–7]. In particular, the boundary value problem of discrete beam equation

$$\left.\begin{array}{l}\Delta^4 y_i = \lambda a_{i+2}y_{i+2}, \quad -1 \leqslant i \leqslant n-2,\\ y_0 = \Delta^2 y_{-1} = \Delta y_n = \Delta^3 y_{n-1} = 0\end{array}\right\} \tag{6}$$

was studied in [7] and a comparison theorem of all of its eigenvalues was established as the sequence $\{a_i\}_{i=1}^n$ varies. Following a similar path, we will study the structure of the spectrum of the boundary value problem of discrete beam Eq. (5). Comparison theorems for all of its eigenvalues will be obtained.

2. The structure of the eigenvalues

In this section, we denote by x^* the conjugate transpose of a vector x. A hermitian matrix A is said to be positive semidefinite if $x^*Ax \geqslant 0$ for any x. It is said to be positive definite if $x^*Ax > 0$ for any nonzero x. In what follows we will write $X \geqslant Y$ if X and Y are hermitian matrices of order n and $X - Y$ is positive semidefinite. A matrix is said to be positive if every component of the matrix is positive.

The boundary conditions in Eq. (5) are the same as

$$y_0 = y_1, \quad y_{-1} = y_2, \quad y_n = y_{n+1}, \quad \text{and} \quad y_{n-1} = y_{n+2}. \tag{7}$$

And the problem in Eq. (5) is equivalent to the linear system

$$(-D - B + \lambda A)y = 0, \tag{8}$$

where

$$A = \operatorname{diag}(a_1, a_2, \ldots, a_{n-1}, a_n),$$
$$B = \operatorname{diag}(b_1, b_2, \ldots, b_{n-1}, b_n),$$
$$y = (y_1, y_2, \ldots, y_{n-1}, y_n)^T$$

and D is a banded $n \times n$ matrix given by

$$D = \begin{pmatrix} 2 & -3 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ -3 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0\\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0\\ 0 & 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 & 1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 & -3\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -3 & 2 \end{pmatrix}.$$

Obviously, there is a one-to-one correspondence between the solutions $(y_1, y_2, \ldots, y_n)^T$ to the problem Eq. (8) and the solutions $\{y_i\}_{i=-1}^{n+2}$ to the problem Eq. (5) under the relationship Eq. (7). In that sense, these two problems are equivalent. We will not distinguish one from the other, denote by y a solution to either one of the two problems in the remainder of this paper.