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Embedding long cycles in faulty k-ary 2-cubes ☆

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ABSTRACT

The class of k-ary n-cubes represents the most commonly used interconnection topology for distributed-memory parallel systems. Given an even $k \geqslant 4$, let (V_1, V_2) be the bipartition of the k-ary 2-cube, f_{v1}, f_{v2} be the numbers of faulty vertices in V_1 and V_2, respectively, and f_e be the number of faulty edges. In this paper, we prove that there exists a cycle of length $k^2 - 2\max\{f_{v1}, f_{v2}\}$ in the k-ary 2-cube with $0 \leqslant f_{v1} + f_{v2} + f_e \leqslant 2$. This result is optimal with respect to the number of faults tolerated.

1. Introduction

The k-ary n-cube Q_n^k has many desirable properties, such as ease of implementation, low-latency and high-bandwidth inter-processor communication. Therefore, a number of distributed-memory parallel systems (also known as multicomputers) have been built with a k-ary n-cube forming the underlying topology. The graph embedding is a technique that maps a guest graph into a host graph (usually an interconnection architecture). Many graph embeddings take cycles and paths as guest graphs [1,3–10], because these interconnection architectures are widely used in distributed-memory parallel systems.

As failures are inevitable, fault-tolerance is an important issue in the distributed-memory parallel system. Recently, fault-tolerant cycle-embeddings of various interconnection networks received much attention (see, for example, [1,3,9]). In [11], Yang et al. proved that the faulty k-ary 2-cube with odd $k \geqslant 3$ admits a hamiltonian cycle if the number of faults does not exceed 2. For even $k \geqslant 4$, Stewart and Xiang [7] investigated the problem of embedding long paths in k-ary 2-cubes with faulty vertices and edges and presented the following result.

Theorem 1 [7]. *Let $k \geqslant 4$ be even, and let f_v be the number of faulty vertices and f_e be the number of faulty edges in Q_2^k with $0 \leqslant f_v + f_e \leqslant 2$. Given any two healthy vertices s and t of Q_2^k, then there is a path from s to t of length at least $k^2 - 2f_v - 1$ in the faulty Q_2^k if s and t have different parities (the parity of a vertex $v = v_1v_2$ of Q_2^k is defined to be $v_1 + v_2$ modulo 2).*

As every pair of adjacent vertices have different parities when k is even, there is a cycle of length at least $k^2 - 2f_v$ in the faulty Q_2^k. In fact, this result can be improved according to the possible distribution of the faulty vertices. The parity of a vertex $v = v_1v_2$ of Q_2^k is defined to be $v_1 + v_2$ modulo 2. We speak of a vertex as being odd or even according to whether its parity is odd or even. In this paper, we prove that there exists a cycle of length $k^2 - 2\max\{f_{v1}, f_{v2}\}$ in the Q_2^k with at most two faults, where f_{v1} (resp. f_{v2}) is the number of faulty vertices which are even (resp. odd). As $f_{v1} + f_{v2} = f_v$, we have $\max\{f_{v1}, f_{v2}\} < f_v$ when $f_{v1} = f_{v2} = 1$. Obviously, $k^2 - 2\max\{f_{v1}, f_{v2}\} \geqslant k^2 - 2f_v$. Therefore, our result improves the result noted above.

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The rest of this paper is organized as follows. In Section 2, we introduce some basic definitions. In Section 3, we prove the main result. Conclusions are covered in Section 4.

2. Basic definitions

Throughout this paper, notation and terminology mostly follow [2].

The k-ary 2-cube Q_2^k is a graph consisting of k^2 vertices, each has the form $v = v_1v_2$, where $0 \leqslant v_1, v_2 \leqslant k-1$. Two vertices $v = v_1v_2$ and $u = u_1u_2$ are adjacent if and only if there exists an integer j, $j \in \{1,2\}$, such that $u_j = v_j \pm 1 \pmod k$ and $u_i = v_i$, for $i \in \{1,2\} \setminus \{j\}$. For clarity of presentation, we omit writing "(mod k)" in similar expressions for the remainder of the paper. A k-ary 2-cube with even $k \geqslant 4$ is a bigraph. Let V_1 (resp. V_2) be the set of the vertices which are even (resp. odd). Then (V_1, V_2) is a bipartition of the k-ary 2-cube. Many structural properties of k-ary 2-cubes are known, but of particular relevance for us is that a k-ary 2-cube is vertex-transitive, that is, for any two distinct vertices u and v of Q_2^k, there is an automorphism of Q_2^k mapping u to v. In particular, the mapping $\theta : ij \to i(k-j)$, $0 \leqslant i, j \leqslant k-1$, is an automorphism of Q_2^k.

For convenience, we write $v_{a,b}$ as the vertex of Q_2^k with the form ab, where $0 \leqslant a, b \leqslant k-1$. For $0 \leqslant i \leqslant j \leqslant k-1$, Row$(i:j)$ is the subgraph of Q_2^k induced by $\{v_{a,b} : i \leqslant a \leqslant j, 0 \leqslant b \leqslant k-1\}$. We simply write Row$(i)$ instead of Row$(i:i)$. It can be seen that each Row(i) is a cycle of length k. Let $(v_{i,j}, v_{i,j+1})$ be an edge of Row(i). Then the edge $(v_{m,j}, v_{m,j+1})$, $m \in \{i-1, i+1\}$, is called the corresponding edge of $(v_{i,j}, v_{i,j+1})$ in Row(m).

3. Cycle embeddings in faulty k-ary 2-cubes

To show our main result, we first introduce some useful lemmas. A pair of vertices $\{u, v\}$ is odd (resp. even) if u and v have different (resp. the same) parities.

According to the proof of Lemma 1 in [7], the following lemma holds.

Lemma 3.1 [7]. *Given an even $k \geqslant 4$, let v be a faulty vertex of Row(0:1) in Q_2^k and let s and t be two distinct healthy vertices of Row(0:1). If $\{v,s\}$ is odd and $\{s,t\}$ is even, then there is a path from s to t of length $2k-2$ in Row(0:1).*

According to Theorem 1, the following lemma holds.

Lemma 3.2 [7]. *Given an even $k \geqslant 4$, let s and t be any two distinct healthy vertices of Q_2^k with two faulty vertices. Then there is a path of length at least $k^2 - 5$ from s to t if $\{s,t\}$ is odd.*

Lemma 3.3 [7]. *Given an even $k \geqslant 4$, let s and t be two distinct healthy vertices of Row$(0:p-1)$ in Q_2^k, where $2 \leqslant p \leqslant k$. If $\{s,t\}$ is odd (resp. even), then there is a path from s to t of length $pk-1$ (resp. $pk-2$) in Row$(0:p-1)$.*

A matching in a graph is a set of pairwise nonadjacent edges. The vertex incident with an edge of a matching is said to be covered by the matching. A perfect matching is one which covers every vertex of the graph. Let G_1 and G_2 be two graphs. We denote by $G_1 \triangle G_2$ the graph induced by the edges of $E(G_1) \triangle E(G_2)$, where $E(G_1) \triangle E(G_2)$ denotes the symmetric difference of $E(G_1)$ and $E(G_2)$. Given an integer m with $1 \leqslant m \leqslant k$, let $M = \{(v_{i,j_1}, v_{i,j_1+1}), (v_{i,j_2}, v_{i,j_2+1}), \ldots, (v_{i,j_m}, v_{i,j_m+1}) : 0 \leqslant j_l \leqslant k-1, l = 1, 2, \ldots, m\} \subseteq E(\mathrm{Row}(i))$ and let $C_{j_n} = (v_{i,j_n}, v_{i+1,j_n}, v_{i+1,j_n+1}, v_{i,j_n+1}, v_{i,j_n})$. Set $\mathcal{C}(M) = C_{j_1} \triangle C_{j_2} \triangle \ldots \triangle C_{j_m}$.

Lemma 3.4. *Given an even $k \geqslant 4$, let v and w be two distinct faulty vertices of Row(0:1) in Q_2^k. If $\{v,w\}$ is odd, then there is a cycle of length $2k-2$ in Row(0:1) that contains at least one healthy edge of Row(0).*

Proof. Without loss of generality, assume that $v = v_{0,0}$. We distinguish two cases.

Case 1. w is a vertex of Row(0). Let $w = v_{0,i} (1 \leqslant i \leqslant k-1)$. As $\theta : v_{i,j} \to v_{i,k-j}$ is an automorphism of Q_2^k, $\theta(v_{0,0}) = v_{0,0}$ and $\theta(v_{0,i}) = v_{0,k-i}$, it is enough to consider $1 \leqslant i \leqslant \frac{k}{2}$. As $\{v,w\}$ is odd and $v_{0,0}$ is even, i is odd. Let $M = \{(v_{0,1}, v_{0,2}), (v_{0,3}, v_{0,4}), \ldots, (v_{0,i-2}, v_{0,i-1}), (v_{0,i+1}, v_{0,i+2}), \ldots, (v_{0,k-2}, v_{0,k-1})\}$. Then it is easy to see that M is a perfect matching of Row$(0) - \{v,w\}$. Thus, $C = \mathrm{Row}(1) \triangle \mathcal{C}(M)$ is a cycle of length $2k-2$ in Row(0:1). As $|M| = \frac{k-2}{2} \geqslant 1$ and $M \subseteq E(C)$, C contains at least one healthy edge of Row(0).

Case 2. w is a vertex of Row(1). Let $w = v_{1,j} (0 \leqslant j \leqslant k-1)$. As $\{v,w\}$ is odd and $v_{0,0}$ is even, j is even. Suppose that $w = v_{1,0}$, then $C = (v_{0,1}, v_{0,2}, \ldots, v_{0,k-2}, v_{0,k-1}, v_{1,k-1}, v_{1,k-2}, \ldots, v_{1,2}, v_{1,1}, v_{0,1})$ is as required. Suppose that $w \neq v_{1,0}$. Let M_1' be the maximum matching of Row$(1) - w$ such that $(v_{1,0}, v_{1,1}) \in M_1'$ and $(v_{1,k-2}, v_{1,k-1}) \notin M_1'$. Set $M_1 = M_1' \cup \{(v_{1,0}, v_{1,k-1})\}$. Let M_0 be the set of corresponding edges of M_1 in Row(0). Then $C = \mathrm{Row}(1) \triangle \mathcal{C}(E(\mathrm{Row}(0)) - M_0)$ is a cycle of length $2k-2$ in Row(0:1). As $|M_0| = |M_1| < k$ and $(E(\mathrm{Row}(0)) - M_0) \subseteq E(C)$, C contains at least one healthy edge of Row(0). $\square$

Theorem 3.1. *Let $k \geqslant 4$ be even, and let $f_v \leqslant 2$ be the number of faulty vertices in Q_2^k. Then there is a cycle of length $k^2 - 2\max\{f_{v1}, f_{v2}\}$ in the faulty Q_2^k, where f_{v1} (resp. f_{v2}) is the number of faulty vertices which are even (resp. odd) and $f_{v1} + f_{v2} = f_v$.*