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Coupled common fixed point theorems for w^*-compatible mappings in ordered cone metric spaces

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ABSTRACT

We establish coupled coincidence point results for mixed g-monotone mappings under general contractive conditions in partially ordered cone metric spaces over solid cones. We also present results on existence and uniqueness of coupled common fixed points. Our results generalize, extend and unify several well known comparable results in the literature. To illustrate our results and to distinguish them from the earlier ones, we equip the paper with examples.

1. Introduction

Fixed point theorems in metric spaces play a major role to construct methods in mathematics for solving problems in applied mathematics and sciences. So the attraction of metric spaces to a large numbers of mathematicians is understandable. Some generalizations of the notion of a metric space have been proposed by some authors.

Huang and Zhang [1] generalized the concept of a metric space to a cone metric space, replacing the set of real numbers by an ordered Banach space, thus reconsidering the notion of a K-space known earlier (see, e.g., [2]). They obtained some fixed point theorems for mappings satisfying different contractive conditions. Several authors obtained further fixed point results in such spaces (see a review of these results in [3]).

Recently, fixed point theory has developed rapidly in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering. The first result in this direction was given by Ran and Reurings [4, Theorem 2.1] who presented its applications to matrix equation. Subsequently, Nieto and López [5] extended this result for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Agarwal et al. [6], as well as O'Regan and Petrusel [7] studied generalized contractions in partially ordered metric spaces. Order contractions in ordered cone metric spaces were considered in [8–10].

Bhaskar and Lakshmikantham [11] introduced the notion of a coupled fixed point and proved some interesting coupled fixed point theorems for mappings satisfying a mixed monotone property (see further Definitions 2.3 and 2.7). Lakshmikantham and Ćirić [12] introduced the concept of a mixed g-monotone mapping and proved coupled coincidence and coupled common fixed point theorems that extend theorems from [11]. Subsequently, many authors obtained several coupled coincidence and coupled fixed point theorems in ordered metric spaces (see, e.g., [13–15]). These results have a lot of applications, e.g., in

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proving existence of solutions of periodic boundary value problems (e.g., [11,16]) as well as integral equations of particular type (e.g., [10,17–19]).

Recently, Karapinar [20] proved coupled fixed point theorems for nonlinear contractions in ordered cone metric spaces over normal cones without regularity. He considered the continuity of commuting mappings in a whole complete space. Shatanawi [21] proved coupled coincidence fixed point theorems in cone metric spaces over cones which were not necessarily normal. See also the results of Sabetghadam et al. [22], Shatanawi [23], Ding and Li [24], and Aydi et al. [25].

In this paper, an attempt is made to prove coupled common fixed point theorems for mixed g-monotone and w^*-compatible mappings satisfying more general contractive conditions in ordered cone metric spaces over a cone that is only solid (i.e., has a nonempty interior). We furnish examples to demonstrate the validity of the results. Our results improve those of Karapinar [20] and Shatanawi [21] in three senses. Firstly, the considered contractive conditions are more general than earlier ones. Secondly, noncommuting maps are considered. Finally, normality of the cone is not assumed. Our work is an ordered version extension of the results of Abbas et al. [26]. An example is presented when our results can be used in proving the existence of a common coupled fixed point, while the results of [26] cannot.

2. Preliminaries

Let E be a real Banach space with respect to a given norm $\|\cdot\|_E$ and let 0_E be the zero vector of E. A non-empty subset P of E is called a *cone* if the following conditions hold: (i) P is closed and $P \neq \{0_E\}$; (ii) $a,\ b \in \mathbb{R},\ a,\ b \geqslant 0,\ x,\ y \in P \Rightarrow ax + by \in P$; (iii) $x \in P,\ -x \in P \Rightarrow x = 0_E$.

Given a cone $P \subset E$, a partial ordering $\leqslant_P$ with respect to P is naturally defined by $x \leqslant_P y$ if and only if $y - x \in P$, for $x, y \in E$. We shall write $x <_P y$ to indicate that $x \leqslant_P y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

The cone P is said to be *normal* if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$0_E \leqslant_P x \leqslant_P y \Rightarrow \|x\|_E \leqslant K\|y\|_E.$$

The least positive number K satisfying the above statement is called the normal constant of P.

In what follows we always suppose that E is a real Banach space with cone P satisfying $\operatorname{int} P \neq \emptyset$ (such cones are called *solid*).

Definition 2.1 [1]. Let X be a non-empty set and $d : X \times X \to P$ satisfies

(i) $d(x,y) = 0_E$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii) $d(x,y) \leqslant_P d(x,z) + d(z,y)$ for all $x, y, z \in E$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.

The concept of a cone metric space is obviously more general than that of a metric space.

Definition 2.2 [1]. Let (X,d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for every $c \in E$ with $0_E \ll_P c$, there is $N \in \mathbb{N}$ such that $d(x_n,x) \ll_P c$ for all $n \geqslant N$, then $\{x_n\}$ is said to converge to x. This limit is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(ii) If for every $c \in E$ with $0_E \ll_P c$, there is $N \in \mathbb{N}$ such that $d(x_n,x_m) \ll_P c$ for all $n, m > N$, then $\{x_n\}$ is called a Cauchy sequence in X.
(iii) If every Cauchy sequence in X is convergent in X, then (X,d) is called a complete cone metric space.

Let (X,d) be a cone metric space. Then the following properties are often used (particularly when dealing with cone metric spaces in which the cone need not be normal):

(p_1) If E is a real Banach space with a cone P and if $a \leqslant_P ha$ where $a \in P$ and $h \in [0,1)$, then $a = 0_E$;
(p_2) if $0_E \leqslant_P u \ll c$ for each $0_E \ll c$, then $u = 0_E$;
(p_3) if $u, v, w \in E$, $u \leqslant_P v$ and $v \ll w$, then $u \ll w$;
(p_4) if $c \in \operatorname{int}P$, $0 \leqslant_P a_n \in E$ and $a_n \to 0_E$, then there exists $k \in \mathbb{N}$ such that for all $n > k$ we have $a_n \ll c$.

Definition 2.3 (*[11,12]*). Let X be a nonempty set and $F : X \times X \to X$, $g : X \to X$. An element $(x,y) \in X \times X$ is called:

(C_1) a coupled fixed point of F if $x = F(x,y)$ and $y = F(y,x)$;
(C_2) a coupled coincidence point of mappings F and g if $gx = F(x,y)$ and $gy = F(y,x)$, and in this case (gx,gy) is called a coupled point of coincidence;
(C_3) a common coupled fixed point of mappings F and g if $x = gx = F(x,y)$ and $y = gy = F(y,x)$.