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Bipanconnectivity of faulty hypercubes with minimum degree ☆

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ABSTRACT

In this paper, we consider the conditionally faulty hypercube Q_n with $n \geqslant 2$ where each vertex of Q_n is incident with at least m fault-free edges, $2 \leqslant m \leqslant n-1$. We shall generalize the limitation $m \geqslant 2$ in all previous results of edge-bipancyclicity. We also propose a new edge-fault-tolerant bipanconnectivity called k-edge-fault-tolerant bipanconnectivity. A bipartite graph is *k-edge-fault-tolerant bipanconnected* if $G-F$ remains bipanconnected for any $F \subset E(G)$ with $|F| \leqslant k$. For every integer m, under the same hypothesis, we show that Q_n is $(n-2)$-edge-fault-tolerant edge-bipancyclic and bipanconnected, and the results are optimal with respect to the number of edge faults tolerated. This not only improves some known results on edge-bipancyclicity and bipanconnectivity of hypercubes, but also simplifies the proof.

1. Introduction

The graph-embedding problem is important when evaluating a network. The graph-embedding problem asks whether the guest graph is a subgraph of a host graph. An embedding strategy provides a scheme to emulate a guest graph on a host graph. This problem has been the subject of many studies in recent years.

Finding a cycle of a given length in graph G is a cycle-embedding problem, and finding cycles of all lengths from 3 to $|V(G)|$ is a pancyclic problem, which is investigated in many interconnection networks [1,2,4,8,11,14,17–22,24,25,29,31]. In general, a graph is *pancyclic* if it contains cycles of all lengths [4]. Pancyclicity is an important property to determine if a network's topology is suitable for an application where mapping cycles of any length into the topology of the network is required. The concept of pancyclicity has been extended to vertex-pancyclicity [16] and edge-pancyclicity [2]. A graph is *vertex-pancyclic* (*edge-pancyclic*) if every vertex (edge) lies on a cycle of every length from 3 to $|V(G)|$. Bipancyclicity is essentially a restriction of the concept of pancyclicity to bipartite graphs whose cycles are necessarily of even length.

Similarly, to find a path between any two distinct vertices $\mathbf{x}$ and $\mathbf{y}$ with a given length in graph G is a path embedding problem; that is, to find a path of length l joining two different vertices $\mathbf{x}$ to $\mathbf{y}$ with $d(\mathbf{x},\mathbf{y}) \leqslant l \leqslant |V(G)|-1$. It is easily noted that any bipartite graph with at least three vertices is not panconnected [28]. For this reason, we say a bipartite graph is *bipanconnected* if there exists a path of length l joining any two different vertices $\mathbf{x}$ and $\mathbf{y}$ with $d(\mathbf{x},\mathbf{y}) \leqslant l \leqslant |V(G)|-1$ such that $2|(l-d(\mathbf{x},\mathbf{y}))$, where expression $2|(l-d(\mathbf{x},\mathbf{y}))$ means that $l-d(\mathbf{x},\mathbf{y}) \equiv 0 \pmod 2$.

Based on this definition, clearly, if *an equitable bipartite graph* with bipartition (X,Y) (i.e. $|X| = |Y|$) is bipanconnected, then it is definitely edge-bipancyclic. Similarly, if a graph is edge-bipancyclic, then it is vertex-bipancyclic. Moreover, if a graph is vertex-bipancyclic, then it is bipancyclic. However, all the converses are not true, as shown in Fig. 1. Therefore, the bipanconnected property is not only more important but also stronger than the other properties.

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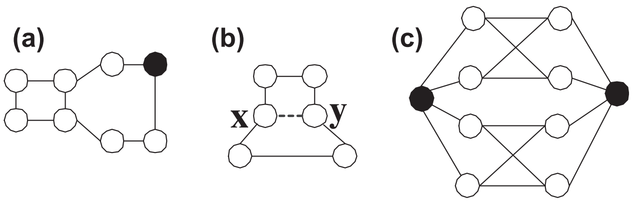

Fig. 1. Clearly, all three graphs are bipartite. (a) The graph is bipancyclic, but no cycle of length 4 contains the black vertex; thus, it is not vertex-bipancyclic. (b) The graph is vertex-bipancyclic. Clearly, no cycle of length 6 contains the edge $(\mathbf{x},\mathbf{y})$; thus, it is not edge-bipancyclic. (c) The graph is edge-bipancyclic. However, no Hamiltonian path joining two black vertices exists. Consequently, it is not bipanconnected.

Fault-tolerant is also a desirable feature in massive parallel systems that have a relatively high probability of failure. A number of fault-tolerant designs for specific multiprocessor architectures have been proposed based on graph-theoretic models, in which the processor-to-processor interconnection structure is represented by a graph.

A bipartite graph G is *k-edge-fault-tolerant edge-bipancyclic* if $G-F$ remains edge-bipancyclic for any $F \subset E(G)$ with $|F| \leqslant k$ [24]. We proposed a new edge-fault-tolerant bipanconnectivity called k-edge-fault-tolerant bipanconnectivity that combines the concepts of k-edge-fault-tolerant edge-bipancyclicity with bipanconnectivity. A bipartite graph is *k-edge-fault-tolerant bipanconnected* if $G-F$ remains bipanconnected for any $F \subset E(G)$ with $|F| \leqslant k$. More precisely, there exists a path of length l joining any two different vertices $\mathbf{x}$ and $\mathbf{y}$ with $d_{G-F}(\mathbf{x},\mathbf{y}) \leqslant l \leqslant |V(G)|-1$ such that $2|(l-d_{G-F}(\mathbf{x},\mathbf{y}))$.

The hypercube is one of the most versatile and unique interconnection networks discovered to date for parallel computation [23,32]. Embedding has been the subject of intensive study, with the hypercube being the host graph and various graphs being the guest graph. The problem of fault-tolerant embedding in the hypercube has been previously studied in [5–7,9,10,12,13,15,17,19,24,26,27,31]. Li et al. [24] proved that for $n \geqslant 2$, Q_n is bipanconnected and $(n-2)$-edge-fault-tolerant edge-bipancyclic. In this paper, we obtain the following results, which are not only an extension of the recent result of Li et al., but also a characterization of the fault distance of Q_n.

Theorem 1. *The n-dimensional hypercube Q_n with $n \geqslant 2$ is $(n-2)$-edge-fault-tolerant bipanconnected.*

Theorem 2. *Let F be an edge subset of Q_n with $|F| \leqslant n-2$. For any two distinct vertices $\mathbf{x}$ and $\mathbf{y}$ of Q_n, there exists a path in $Q_n - F$ of length l with*

$$\begin{cases} d(\mathbf{x},\mathbf{y}) \leqslant l \leqslant 2^n - 1 & \text{and} \quad 2|(l-d(\mathbf{x},\mathbf{y})), \ \text{if} \ |F| < d(\mathbf{x},\mathbf{y}), \\ d(\mathbf{x},\mathbf{y})+2 \leqslant l \leqslant 2^n - 1 & \text{and} \quad 2|(l-d(\mathbf{x},\mathbf{y})), \ \text{if} \ |F| \geqslant d(\mathbf{x},\mathbf{y}). \end{cases}$$

From the definition, by Theorem 1, the following proof is straightforward.

Corollary 1 [24]. *For $n \geqslant 2$, Q_n is $(n-2)$-edge-fault-tolerant edge-bipancyclic.*

A graph G is *conditionally faulty* if each vertex is incident with at least m fault-free edges, $2 \leqslant m \leqslant \delta(G)-1$. Let F be an edge subset of G with $|F| \leqslant k$. The *conditionally edge-fault-tolerant edge-bipancyclicity*, $\mathscr{C}_m(G)$, is defined to be the maximum integer k such that a conditionally faulty bipartite graph G with $\delta(G-F) \geqslant m$ is k-edge-fault-tolerant edge-bipancyclic, and undefined otherwise. Similarly, the *conditionally edge-fault-tolerant bipanconnectivity*, $\mathscr{P}_m(G)$, is defined to be the maximum integer k such that a conditionally faulty equitable bipartite graph G with $\delta(G-F) \geqslant m$ is k-edge-fault-tolerant bipanconnected, and undefined otherwise.

Theorem 3. *For every integer $m, \mathscr{C}_m(Q_n) = \mathscr{P}_m(Q_n) = n-2$ if $n \geqslant 2$.*

The remainder of this paper is organized as follows. The next Section introduces some basic definitions and related works. In Section 3, we prove the main results. Finally, Section 4 provides the conclusions.

2. Preliminaries

In this paper, graph-theoretical terminology and notation in [3] are used, and a graph $G=(V,E)$ means a simple graph, where $V=V(G)$ is the vertex set and $E=E(G)$ is the edge set of the graph G. For a vertex $\mathbf{u}$, $N_G(\mathbf{u})$ denotes the *neighborhood* of $\mathbf{u}$, which is the set $\{\mathbf{v} | (\mathbf{u},\mathbf{v}) \in E\}$. And $|N_G(\mathbf{u})|$ is the *degree* of $\mathbf{u}$, denoted by $d_G(\mathbf{u})$. Moreover, *the minimum degree* of G, denoted by $\delta(G)$, is $\min\{d_G(\mathbf{v}) | \mathbf{v} \in V(G)\}$. Two vertices $\mathbf{u}$ and $\mathbf{v}$ are *adjacent* if $(\mathbf{u},\mathbf{v}) \in E$. A graph $P = \langle \mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_k} \rangle$ is called a *path* if $k+1$ vertices $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_k}$ are distinct and $(\mathbf{v_{i-1}}, \mathbf{v_i})$ is an edge of P for $i = 1,2,\ldots,k$. Two vertices $\mathbf{v_0}$ and $\mathbf{v_k}$ are called *end-vertices* of the path, and the number k of the edges contained in the path is called the *length* of P, denoted by $l(P)$. For convenience, we use the sequence $P = \langle \mathbf{v_0}, \ldots, \mathbf{v_i}, P[\mathbf{v_i}, \mathbf{v_j}], \mathbf{v_j}, \ldots, \mathbf{v_k} \rangle$, where $P[\mathbf{v_i}, \mathbf{v_j}] = \langle \mathbf{v_i}, \mathbf{v_{i+1}}, \ldots, \mathbf{v_j} \rangle$ and the two vertices $\mathbf{v_i}$