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On the coupling of LDG-FEM and BEM methods for the three dimensional magnetostatic problem

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ABSTRACT

We present two new coupling models for the three dimensional magnetostatic problem. In the first model, we propose a new coupled formulation, prove that it is well posed and solves Maxwell's equations in the whole space. In the second, we propose a new coupled formulation for the Local Discontinuous Galerkin method, the finite element method and the boundary element method. This formulation is obtained by coupling the LDG method inside a bounded domain Ω_1 with the FEM method inside a layer $\Omega_2 := \Omega \setminus \overline{\Omega}_1$ where Ω is a bounded domain which is made up of material of permeability μ and such that $\overline{\Omega}_1 \subset \Omega$, and with a boundary element method involving Calderon's equations. We prove that our formulation is consistent and well posed and we present some a priori error estimates for the method.

1. Introduction

In this work, we study Maxwell's equations for magnetostatic problem in IR^3. We consider a bounded domain Ω which is made up of material of permeability μ. Outside the domain Ω, in Ω^c, we assume that the permeability μ is a constant and the value of μ in Ω^c is the permeability in vacuum. Inside Ω, we suppose that μ is a function of $x = (x_1,x_2,x_3) \in \mathrm{IR}^3$ which is in $L^\infty(\Omega)$, and $\mu(x) \geqslant c > 0$ for all $x \in \Omega_1$. For the current density J, we suppose that it has a bounded support included in Ω, and is in $H_0(\mathrm{div}^0, \Omega)$ (for the definition of this space, see the next section). We study the following problem: for a given field J, find the magnetic field h which solves the following magnetostatic problem model derived from Maxwell's equations

$$\begin{cases} \mathrm{rot}\ h = J, & \text{in } \mathrm{IR}^3, \\ \mathrm{div}\ \mu h = 0, & \text{in } \mathrm{IR}^3. \end{cases} \tag{1.1}$$

The outline of this paper is as follows. In the next section some notations and general results are presented which are necessary for the study of our model problem (1.1). In Section 3, by introducing a scalar potential and a vector potential, we establish a variational formulation associated to the continuous problem, prove that it is well posed and show the equivalence to the model problem. In Section 4 we present our coupled LDG-FEM and BEM formulation. For the LDG method, we introduce auxiliary variables. The coupling of the LDG with FEM methods is introduced by using numerical fluxes. We prove that our formulation is consistent and well posed. We finish this section by giving a primal formulation which we use in Section 5 to establish error estimates. An h-version error analysis is carried out in Section 5 and concluding remark are presented in Section 6.

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2. Preliminaries and notations

Given a domain D in IR^2 or IR^3, we denote by $H^s(D)^d$, d = 1, 2, 3, the Sobolev space of real valued functions with integer or fractional regularity exponent $s \geqslant 0$, endowed with the norm $\|\cdot\|_{s,D}$; see, e.g., [10] for details.

For $D \subset \mathrm{IR}^3$, $H(\mathrm{rot},D)$ and $H(\mathrm{div},D)$ are the spaces of real valued vector functions $u \in L^2(D)^3$ with $\mathrm{rot} u \in L^2(D)^3$ and $\mathrm{div} u \in L^2(D)$, respectively, endowed with the graph norms. We denote by $H_0^1(D)$, $H_0(\mathrm{rot},D)$, $H_0(\mathrm{div},D)$ the subspaces of $H^1(D)$, $H(\mathrm{rot},D)$, $H(\mathrm{div},D)$ of functions with zero trace, tangential trace and normal trace on ∂D, respectively. The spaces $H(\mathrm{rot}^0,D)$ and $H(\mathrm{div}^0,D)$ are the subspaces of $H(\mathrm{rot},D)$ and $H(\mathrm{div},D)$ consisting of irrotational and *divergence*-free functions, respectively. Let $(\cdot,\cdot)$ denote the scalar product on $L^2(D)$ or $L^2(D)^3$.

If $\Gamma = \partial D$, we define

$$H^{\frac{1}{2}}(\Gamma) := \left\{ v \in L^2(\Gamma), \|v\|_{H^{\frac{1}{2}}(\Gamma)} < \infty \right\},$$

where

$$\|v\|^2_{H^{\frac{1}{2}}(\Gamma)} := \frac{1}{diam(D)} \int_{\partial D} |v(x)|^2 \, ds_x + \int_{\partial D} \int_{\partial D} \frac{|v(x) - v(y)|^2}{|x-y|^2} \, ds_x ds_y \tag{2.1}$$

is the corresponding norm. See [1].

We denote by $H^{-\frac{1}{2}}(\Gamma)$ the dual space of $H^{\frac{1}{2}}(\Gamma)$ and by $\langle,\rangle$ the duality pairing between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$. More generally, if Γ is a proper (non-empty) open subset of ∂D, we define $H_{00}^{\frac{1}{2}}(\Gamma)$ as the subspace of $H^{\frac{1}{2}}(\Gamma)$ consisting those functions defined on Γ and whose extension by zero on $\partial D \backslash \Gamma$ belongs to $H^{\frac{1}{2}}(\partial D)$. Let us recall that it is coincide with $H^{\frac{1}{2}}(\Gamma)$ provided that Γ is a surface without boundary; see e.g., [2]. The space $H_{00}^{-\frac{1}{2}}(\Gamma)$ is the dual space of $H_{00}^{\frac{1}{2}}(\Gamma)$.

2.1. Variational framework

Throughout this paper, Ω will denote a bounded Lipschitz polyhedron included in IR^3 which is supposed to be both connected and simply connected; in particular we suppose that Ω is such that $H(\mathrm{rot},\Omega) \cap H_0(\mathrm{div},\Omega) \hookrightarrow H^1(\Omega)^3$. Γ is the boundary of Ω which is also assumed to be sufficiently smooth, connected and simply connected, and n is the unit outward normal on Γ; we also set $\Omega' := \mathrm{IR}^3 \setminus \overline{\Omega}$.

2.2. Integral operators and Calderon's equations

Now we turn to the study of the properties of integral operators which will be involved in the boundary integral method.

The integral operators V, K and W denote the single layer potential, the double layer potential and the hypersingular operator, respectively, and are defined by:

$$Ku(x) = \int_\Gamma u(y) \frac{\partial}{\partial n_y} E(x,y) ds_y \quad \forall u \in H^{\frac{1}{2}}(\Gamma),$$
$$Vu(x) = \int_\Gamma u(y) E(x,y) ds_y \quad \forall u \in H^{\frac{-1}{2}}(\Gamma),$$

and

$$Wu(x) = -\frac{\partial}{\partial n_x} \int_\Gamma u(y) \frac{\partial}{\partial n_y} E(x,y) ds_y \quad \forall u \in H^{\frac{1}{2}}(\Gamma).$$

Here, $E(x,y) = (4\pi|x-y|)^{-1}$ is the fundamental solution for the three dimensional Laplacian problem and $\frac{\partial}{\partial n_y}$ denotes the weak derivate with respect to the variable y.

We have the following properties for these operators.

Lemma 2.1. *The operators previously defined satisfy:*

$$V : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma),$$
$$K : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma),$$
$$K' : H^{\frac{-1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma),$$
$$W : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma),$$

where K' is the adjoint operator of K. Furthermore, all four operators are linear and continuous. V is symmetric, self-adjoint and positive definite. W is symmetric, self-adjoint and positive semidefinite provided that the capacity of Γ is smaller than 1 which is assumed here.