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Two-grid methods for characteristic finite volume element solution of semilinear convection–diffusion equations ☆

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ABSTRACT

Two-grid methods for characteristic finite volume element solutions are presented for a kind of semilinear convection-dominated diffusion equations. The methods are based on the method of characteristics, two-grid method and the finite volume element method. The nonsymmetric and nonlinear iterations are only executed on the coarse grid (with grid size H). And the fine-grid solution (with grid size h) can be obtained by a single symmetric and linear step. It is proved that the coarse grid can be much coarser than the fine grid. The two-grid methods achieve asymptotically optimal approximation as long as the mesh sizes satisfy $H = O(h^{1/3})$.

1. Introduction

Convection–diffusion transport *partial differential equations* (PDEs) arise in petroleum reservoir simulation, subsurface contaminant transport, and many other important applications. We are concerned with the numerical approximation of the solutions of convection–diffusion problems in which the convection or transport dominates the diffusion. In this paper, we shall consider combining the *modified method of characteristics* (MMOC) and two-grid method with finite volume element method to treat the convection–diffusion problems given by

$$\begin{cases} c(x)\frac{\partial u}{\partial t} + \mathbf{b}(x)\cdot\nabla u - \nabla\cdot(a(x)\nabla u) = f(u,x,t), & (x,t)\in\Omega\times(0,T],\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega = (a,b)\times(c,d)$ is a rectangular domain, $x = (x_1,x_2)\cdot\mathbf{b}(x) = (b_1(x),b_2(x))^{\mathrm{T}}$ and $T > 0$ is some fixed final time. For convenience, we assume that problem (1.1) is Ω-periodic, that is, all functions of (1.1) are spatially Ω-periodic [1–3]. Throughout this paper we assume the coefficients of (1.1) satisfy

(a) $0 < a_* \leqslant a(x) \leqslant a^*, |\mathbf{b}| = \sqrt{b_1^2 + b_2^2} \leqslant b^*, 0 < c_* \leqslant c(x) \leqslant c^*$;
(b) $\left|\frac{\mathbf{b}(x)}{c(x)}\right| + \left|\frac{\partial}{\partial x_i}\left(\frac{\mathbf{b}(x)}{c(x)}\right)\right| \leqslant K_1$;
(c) $f(u,x,t)$ holds uniformly Lipschitz condition with respect to u and $\left|\frac{\partial f}{\partial x_i}\right| + \left|\frac{\partial f}{\partial u}\right| + \left|\frac{\partial^2 f}{\partial u^2}\right| \leqslant K_2,\ i = 1,2$, where a_*, a^*, b^*, c_*, c^*, K_1 and K_2 are positive constants. We also assume that the solution u of (1.1) satisfies
(d) $u \in L^\infty(0,T; W^{q,2}(\Omega)\cap W^{2,\infty}(\Omega))$;

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(e) $\frac{\partial u}{\partial t} \in L^2(0,T;W^{q,2}(\Omega)) \cap L^\infty(0,T;W^{q,2}(\Omega))$;
(f) $\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;L^2(\Omega))$,

for some $q \geqslant 2$.

The MMOC was first formulated for convection–diffusion equations by Douglas and Russell in [4]. And then extended by Russell [5] to nonlinear coupled systems in two and three spatial dimensions. In the MMOC the time derivative and the convection term are combined as a directional derivative along the characteristics, leading to a characteristic time-stepping procedure. Consequently, the MMOC stabilizes the governing PDEs, allowing for large time steps in a simulation without loss of accuracy, and eliminates the excessive numerical dispersion and grid orientation effects present in many upwind methods [4].

Two-grid method was first introduced by Xu [6,7] as a discretization method for linear (nonsymmetric or indefinite) and especially nonlinear elliptic partial equations. The basic idea of this method is to solve a complicated problem (nonlinear, nonsymmetric indefinite, etc.) on a coarse grid (with mesh size H) and then solve an easier problem (linear, SPD, etc.) on a fine grid (with mesh size h and $h \ll H$) as correction. Later on, the two-grid method was further investigated by many authors [8–10]. Dawson and Wheeler [8,9] applied this method combined with the mixed finite element method and the finite difference method to a kind of parabolic problems; Li and Allen [10] have applied two-grid method combined with mixed finite element method to reaction–diffusion equations; Chen, Huang et al., [11] have constructed a two-grid method for expanded mixed finite element solution of semilinear reaction–diffusion equations.

Finite volume element (FVE) method, as a type of important numerical tool for solving differential equations, was widely used in several engineering fields, such as fluid mechanics, heat and mass transfer and petroleum engineering. Perhaps the most important property of FVE method is that it can preserve the conservation laws (mass, momentum, heat flux) on each computational cell. This important property, combined with adequate accuracy and ease of implementation, has attracted many researchers to do research. The theoretical framework and the basic tools for the analysis of FVE method have been developed in the last two decades (see, e.g. [12–19]).

In this paper, we consider combining the MMOC and two-grid method with finite volume element method to treat problem (1.1). We choose two conforming finite element spaces V_H and V_h on one coarse grid with mesh size H and one fine grid with mesh size $h \ll H$ as the two-grid spaces, respectively. We solve a nonsymmetric and nonlinear problem on the coarse grid space, then we use the known coarse grid solution and a Taylor expansion to extrapolate the solution on the fine grid. On the fine grid we only need to solve a symmetric and linear system. A remarkable fact about this simple approach is, as shown in [6], that the coarse mesh can be quite coarse and still maintain a good accuracy approximation. A brief outline of this paper is as follows. In Section 2, we give the partition and some preliminaries. In Section 3, we describe the characteristic FVE method and the two-grid characteristic FVE method and give two algorithms for the two-grid FVE method. In Section 4, we analyze error estimate of the characteristic FVE method. Section 5 is devoted to the error estimates for the two-grid characteristic FVE method. In the last section, we give a conclusion.

Throughout this paper, let C or with its subscription stand for a generic positive constant which does not depend on the the spatial or time discretization parameters and may be different at its different occurrences.

2. Preliminaries and notation

We will use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ with $1 \leqslant p \leqslant \infty$ consisting of functions that have generalized derivatives of order s in the space $L^p(\Omega)$. The norm of $W^{s,p}(\Omega)$ is defined by

$$\|u\|_{s,p,\Omega} = \|u\|_{s,p} = \left(\int_\Omega \sum_{|\alpha| \leqslant s} |D^\alpha u|^p dx \right)^{\frac{1}{p}}, \quad 1 \leqslant p \leqslant \infty,$$

with the standard modification for $p = \infty$. In order to simplify the notation, we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and omit the index $p = 2$ and Ω whenever possible; i.e., $\|u\|_{s,2,\Omega} = \|u\|_{s,2} = \|u\|_s$.

For the domain Ω, we consider a quasi-uniform regular triangulation T_h consisting of closed triangle elements K such that $\overline{\Omega} = \cup_{K \in T_h} K$. We will use $\mathcal{N}_h$ to denote the set of all nodes or vertices of T_h,

$$\mathcal{N}_h = \{p : p \text{is a vertex of element } K \in T_h \textit{ and } p \in \overline{\Omega}\},$$

and $\mathcal{N}_h^0 = \mathcal{N}_h \cap \Omega$.

Then we introduce a dual mesh T_h^* based on T_h. There are various ways to introduce the dual mesh. Almost all approaches can be described by the following general scheme. In each element $K \in T_h$ consisting of vertices x_i, x_j, x_k, select a point Q in K, and select a point x_{ij} on each of the three edges $\overline{x_i x_j}$ of K. Then connect Q to the points x_{ij} by straight lines r_{ij}. Then for a vertex x_i, we let V_i be the polygon whose edges are r_{ij} in which x_i is a vertex of the element K. We call V_i a control volume centered at x_i. Obviously we have $\cup_{x_i \in \mathcal{N}_h} V_i = \overline{\Omega}$, and the dual mesh T_h^* is then defined as the set of these control volumes.

We call the control volume mesh T_h^* quasi-uniform regular if there exists a positive constant $C > 0$, such that

$$C^{-1}h^2 \leqslant meas(V_i) \leqslant Ch^2, \ \forall V_i \in T_h^*,$$

where h is the maximum diameter of all elements $K \in T_h$.