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Finite element solution of nonlinear diffusion problems

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ABSTRACT

In this paper we describe the Rothe-finite element numerical scheme to find an approximate solution of a nonlinear diffusion problem modeled as a parabolic partial differential equation of even order. This scheme is based on the Rothe's approximation in time and on the finite element method (FEM) approximation in the spatial discretization. A proof of convergence of the approximate solution is given and error estimates are shown.

1. Introduction

At the beginning, we present some notations used during our work. First, the standard notations of functional analysis (see e.g., [2,4,9]) for different spaces $L_2(\Omega)$, $H^1 \equiv W^{1,2}(\Omega)$ and $H_0^1(\Omega) \subseteq V \subseteq H^1(\Omega)$ are used here. The notations $|\cdot|$ and $\|\cdot\|$ represent the norms in $L_2(\Omega)$ and V, respectively. Finally, the values C, ε and C_ε are generic and positive constants independent of the discretization parameters to be introduced below where ε is small and $C_\varepsilon = C(\varepsilon^{-1})$.

In the present paper we are dealing with the numerical approximation of the following parabolic initial value problem. Find u: $Q \equiv \Omega \times [0,T] \to R$ such that

$$\partial_t \beta(u) - \Delta u = f(x,t), \quad \text{in } Q, \tag{1}$$

for $t \in I \equiv [0,T]$ and $x \in \Omega \subset R^D$ $(D = 1,2)$ where Ω is a simply connected and bounded domain with Lipschitz continuous boundary $\partial\Omega \cdot \beta(u)$ is the nonlinear term satisfying the following two cases:

(i) degenerate case $0 \leqslant \beta'(u) \leqslant \infty$,
(ii) nondegenerate case $0 < \varepsilon \leqslant \beta'(u) \leqslant C_\varepsilon < \infty$.

The source function $f(x,t)$ satisfies the Lipschitz condition [1], i.e.,

$$|f(x,t) - f(x,t')| \leqslant C|t - t'|.$$

The Eq. (1) is coupled with boundary and initial conditions of the form

$$u(x,t) = 0, \quad \text{on } \Gamma_1 \times [0,T], \tag{2}$$
$$\nabla u \cdot v = g(x,t), \quad \text{on } \Gamma_2 \times [0,T], \tag{3}$$
$$u(x,0) = u_0(x), \quad \text{in } \Omega \tag{4}$$

where v is the outward normal on Γ_2 and $g(x,t)$ satisfies the Lipschitz condition analogously for f. Here Γ_1 and Γ_2 are open subsets of $\partial\Omega$ such that $\Gamma_1 \cap \Gamma_2 = \Phi$ and $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega$.

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The problem (1)–(4) represents a large class of heat conduction problems, e.g., gas diffusion in porous media, infiltration, and phase change phenomena. Also, it describes the contaminant transport, in its linear form, in porous media intensively studied in the last years. It arises also in modeling the immiscible flow of two fluids in porous medium. For example, sea oil and water flow in petroleum reservoir engineering and air and water flow in agricultural fields (see e.g., [6]).

The weak formulation of the problem (1)–(4) is established here to reduce the order of the parabolic differential Eq. (1) from second order to first one. The weak form is linearized by approximating the nonlinear term $\beta'(u)$ by certain relaxation parameter $\lambda(x,t)$. The spatial domain Ω is discretized by N isoparametric elements Ω^e such that $\Omega = \bigcup_{e=1}^{N}\Omega^e$ and the solution u is approximated on each element by $u^e(t)$. Finally, the Rothe method is used to discretize the time interval $[0,T]$ and to approximate $u^e(t)$ on each element.

2. Weak formulation

At the beginning, we assume that:

(A1) $u_0 \in V$,
(A2) the bilinear form $((\cdot,\cdot))$ is symmetric, bounded and V-elliptic, i.e.,

$$|((u,v))| \leqslant C\|u\|\|v\|, \quad ((u,u)) \geqslant C\|u\|^2 \ \forall\ u,v \in V. \tag{5}$$

(A3) the function $f(x,t)$ satisfies the growth condition

$$|f(x,t,\xi)| \leqslant C(1+|\xi|) \quad \forall (x,t,\xi) \in \Omega \times I \times R$$

and analogously for $g(x,t)$.

Let us now introduce the weak form for the problem (1)–(4) which consists of finding $u \in H^1$ satisfying

$$(\partial_t\beta,\chi_i) - (\Delta u,\chi_i) - (f,\chi_i) = 0, \quad i = 1,2,3,\ldots,n_m, \tag{6}$$

where n_m is the total number of the applied domain mesh nodes and $\chi_i \in V$ is a test function at the mesh node i. Applying Green's theorem on the second term in Eq. (6) and inserting the boundary conditions (2), (3) into Eq. (6) yield

$$(\partial_t\beta,\chi_i) + (\nabla u,\nabla\chi_i) - (f,\chi_i) - (g,\chi_i)_{\Gamma_2} = 0, \tag{7}$$

where $i = 1,2,3,\ldots,n_m$.

3. Spatial discretization by finite elements

The spatial domain Ω is divided into N isoparametric elements without interior points, i.e., $\Omega = \bigcup_{e=1}^{N}\Omega^e$ and $V^{e,p}$ is defined by

$$V^{e,p} = \{\varphi \in H^1 | \varphi(x) \in Q_p(\Omega^e),\ x \in \Omega^e,\ e = 1,2,\ldots,N\}, \tag{8}$$

where $Q_p(\Omega^e)$ is the space of bi-polynomial functions that are products of polynomials of degree p in the spatial coordinates on Ω^e (see e.g., [14]).

Finite element discretization of Eq. (7) and (4) is obtained by approximating H^1 by finite dimensional subspaces $V^{e,p}$ and determining $u^e \in V^{e,p}$ such that

$$(\partial_t\beta^e,\chi_i) + (\nabla u^e,\nabla\chi_i) - (f,\chi_i) - (g,\chi_i)_{\Gamma_2} = 0, \tag{9}$$

$$u^e(x,0) = u_0^e, \tag{10}$$

where $i = 1,2,3,\ldots,n_m$.

For isoparametric element with eight nodes, $i = 1,2,3,\ldots,8$, (see e.g., [3,5,8,13]), the Eq. (9) could be rewritten as follows:

$$(\partial_t\beta^e,\chi_i)_{\Omega^e} + (\nabla u^e,\nabla\chi_i)_{\Omega^e} - (f,\chi_i)_{\Omega^e} - (g,\chi_i)_{\Gamma_2} = 0, \tag{11}$$

where

$$u^e(x) = \sum_{j=1}^{8} u_j(t)\varphi_j(x), \tag{12}$$

$$\partial_t\beta^e = \sum_{j=1}^{8} \left.\frac{d\beta(u)}{dt}\right|_j \varphi_j(x) \tag{13}$$

over the element Ω^e. Notice that φ_j, u_j and $\left.\frac{d\beta(u)}{dt}\right|_j$ are the values of the interpolating functions φ, u and $\frac{d\beta(u)}{dt}$ at the node j, respectively on the element Ω^e where $j = 1,2,3,\ldots,8$. Also, if Ω^edo not has any sides on Γ_2 then the fourth term in Eq. (11) will vanish.