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Convenient analytic recurrence algorithms for the Adomian polynomials

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ABSTRACT

In this article we present four analytic recurrence algorithms for the multivariable Adomian polynomials. As special cases, we deduce the four simplified results for the one-variable Adomian polynomials. These algorithms are comprised of simple, orderly and analytic recurrence formulas, which do not require time-intensive operations such as expanding, regrouping, parametrization, and so on. They are straightforward to implement in any symbolic software, and are shown to be very efficient by our verification using MATHEMATICA 7.0. We emphasize that from the summation expressions, $A_n = \sum_{k=1}^{n} U_n^k$ for the multivariable Adomian polynomials and $A_n = \sum_{k=1}^{n} f^{(k)}(u_0)C_n^k$ for the one-variable Adomian polynomials, we obtain the recurrence formulas for the U_n^k and the C_n^k. These provide a theoretical basis for developing new algorithmic approaches such as for parallel computing. In particular, the recurrence process of one particular algorithm for the one-variable Adomian polynomials does not involve the differentiation operation, but significantly only the arithmetic operations of multiplication and addition are involved; precisely $C_n^1 = u_n$ $(n \geqslant 1)$ and $C_n^k = \frac{1}{n}\sum_{j=0}^{n-k}(j+1)u_{j+1}C_{n-1-j}^{k-1}$ $(2 \leqslant k \leqslant n)$. We also discuss several other algorithms previously reported in the literature, including the Adomian–Rach recurrence algorithm [1] and this author's index recurrence algorithm [23,36].

1. Introduction

The Adomian decomposition method (ADM) and its modifications [1–11] are practical techniques for solving functional equations. The method, which requires neither linearization nor perturbation, efficiently works for a wide class of initial value or boundary value problems, encompassing linear, nonlinear, and even stochastic systems, see e.g. [3–9,11–18].

We introduce the ADM by using the initial value problem for a second order ordinary differential equation in the Adomian form rather than the usual Picard form

$$Lu + Ru + Nu = g(t), \tag{1}$$

where $L = \frac{d^2}{dt^2}$, R is the remaining linear part containing the lower order derivatives, N represents a nonlinear analytic operator, $u(0)$ and $u'(0)$ are the specified initial conditions, and $g(t)$ is a given bounded, analytic function.

Operating with L^{-1}, where L^{-1} is the twofold definite integral operator from 0 to t, on both sides of (1) leads to

$$u = u(0) + u'(0)t + L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{2}$$

The method supposes a decomposition series solution and decomposes the nonlinear term Nu into a series

$$u = \sum_{n=0}^{\infty} u_n, \quad Nu = \sum_{n=0}^{\infty} A_n, \tag{3}$$

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where the A_n, depending on the solution components u_0, u_1, ..., u_n, are called the Adomian polynomials, and are defined for the nonlinearity $Nu = f(u)$ by the definitional formula [1–6]

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[f\left(\sum_{k=0}^{\infty}u_k\lambda^k\right)\right]_{\lambda=0}, \quad n = 0,1,2,\ldots, \tag{4}$$

where λ, the analytic parameter, is simply a grouping parameter of convenience. The first five Adomian polynomials are

$$\begin{aligned}
A_0 &= f(u_0),\\
A_1 &= f'(u_0)u_1,\\
A_2 &= f'(u_0)u_2 + f''(u_0)\tfrac{u_1^2}{2!},\\
A_3 &= f'(u_0)u_3 + f''(u_0)u_1u_2 + f^{(3)}(u_0)\tfrac{u_1^3}{3!},\\
A_4 &= f'(u_0)u_4 + f''(u_0)\left(u_1u_3 + \tfrac{u_2^2}{2!}\right) + f^{(3)}(u_0)\tfrac{u_1^2u_2}{2!} + f^{(4)}(u_0)\tfrac{u_1^4}{4!}.
\end{aligned} \tag{5}$$

The decomposition method consists in identifying the u_n's by means of the recursion scheme

$$u_0 = u(0) + u'(0)t + L^{-1}g, \tag{6}$$

$$u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n, \quad n = 0,1,2,\ldots. \tag{7}$$

The convergence of the method has been extensively discussed in [10,19–23]. In this paper the solution of the equation is viewed as a decomposition of the pre-existent, unique, analytic function, which identically satisfies the mathematical model under consideration to be determined by recursion.

The n-term approximation of the solution, $\phi_n = \sum_{i=0}^{n-1}u_i$, requires the Adomian polynomials A_0, A_1, ..., A_{n-2} in the nonlinear case. The computation of the Adomian polynomials is a key procedure for the method and different algorithms for the Adomian polynomials have been proposed in order to improve computational efficiency [1,10,21–32].

Adomian and Rach [1] gave the first recurrence algorithm for the Adomian polynomials. It can be expressed as

$$A_n = \frac{1}{n!}\sum_{i=1}^{n}f^{(i)}(u_0)\cdot c(i,n)|_{\lambda=0}, \quad n \geqslant 1, \tag{8}$$

with the recurrence rule for $c(i,n)$

$$c(i,j) = c(i-1,j-1)\frac{du_\lambda}{d\lambda} + \frac{dc(i,j-1)}{d\lambda}, \quad 1 \leqslant i \leqslant j, \tag{9}$$

where u_λ represents the analytic parametrization

$$u_\lambda = \sum_{k=0}^{\infty}u_k\lambda^k$$

and letting

$$c(0,0) = 1, \quad c(0,j) = 0 \quad (j \geqslant 1), \quad c(i,j) = 0 \ (i > j). \tag{10}$$

See also [2–4,6]. This recurrence algorithm will be further discussed in Section 4.

Rach [24] gave the first formula discarding the analytic parametrization, which is called Rach's Rule, see Page 16 in [4] and Page 51 in [6],

$$A_n = \sum_{k=1}^{n}f^{(k)}(u_0)C(k,n), \quad n \geqslant 1, \tag{11}$$

where $C(k,n)$ are the sums of all possible products of k components from u_1, u_2, ..., u_n, whose subscripts sum to n, divided by the factorial of the number of repeated subscripts, that is

$$C(k,n) = \sum_{\sum_{j=1}^{n}jp_j=n,\ \sum_{j=1}^{n}p_j=k}\frac{u_1^{p_1}u_2^{p_2}\ldots u_n^{p_n}}{p_1!p_2!\ldots p_n!}. \tag{12}$$

We note that the $c(i,n)$ in Eq. (8) is related to the parameter λ, while the $C(k,n)$ in Eq. (11) does not include λ.

Riganti [25] gave a convenient recurrence formula for A_n, see Corollary 2 in this article. Abbaoui and Cherruault [21,22] denoted the A_n by

$$A_n = \sum_{\sum_{j=1}^{n}jp_j=n} f^{(\sum_{j=1}^{n}p_j)}(u_0)\frac{u_1^{p_1}u_2^{p_2}\ldots u_n^{p_n}}{p_1!p_2!\ldots p_n!}, \quad n > 0 \tag{13}$$

and gave another expression by dividing n into all possible decreasing sequences of nonnegative integers.