Applied Mathematics and Computation 218 (2011) 1553–1567

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Investigation of Nash Equilibrium existence involving complementarity-constrained pricing models

Wanmei Soon

Mathematics and Mathematics Education Academic Group, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616, Singapore

ARTICLE INFO

Keywords:
Multi-product
Pricing models
Complementarity constrained optimization problem
Nash Equilibrium
Non-cooperative game

ABSTRACT

The goal of this paper is to study Nash Equilibrium (NE) existence of some game-theoretic pricing models. In Soon et al. [17], deterministic pricing models incorporating a complete demand system were proposed. As in those models, the demand function is defined via a Nonlinear Complementarity Problem (NCP), the models' pricing constraints include complementarity conditions. When incorporated within a game, the best response problem facing each seller is a Mathematical Program with Equilibrium Constraints. A randomized version of this pricing problem will be introduced in this work and the issue of NE existence will be discussed for both the deterministic and random pricing games.

1. Introduction

Many multi-product pricing models have been introduced in the literature to determine production lines, allocations of resources, choices of product differentiation and so on, in supply chains, revenue management of perishable assets etc. Among the different types of models, the demand for products is either assumed known and fixed, or of some well-known form (e.g., a linear function), or to satisfy certain properties (e.g., concavity). We are interested in the possibility of existence of Nash Equilibrium policies in pricing games which explicitly consider demands as functions of prices. The reader can refer to [16] for a review of relevant models. In particular, in this work, the NE issue for some special forms of the pricing games first seen in Soon et al. [17] is studied.

In the paper [17], it was discussed that many pricing models consider incomplete demand functions which are defined only on a restricted domain. Some examples were presented to show that incomplete demand functions may lead to inferior pricing models. Thus pricing models that incorporate a demand function defined on the entire set of nonnegative prices were proposed in that work. Since in such a model, the demand system involves complementarity constraints, the model is an NCP-constrained optimization problem, or a type of Mathematical Program with Equilibrium Constraints (MPEC). In addition, we have shown that, under certain conditions, the complementarity constraints in the pricing model can be eliminated. However in many cases, these constraints cannot be ignored and remain as a core structure in the model.

When these pricing models are used in a game context, as the best response problem for each seller in the pricing game is itself an MPEC, the game is thus an example of an *Equilibrium Problem with Equilibrium Constraints* (EPEC). In the theses by Su [18] and Ehrenmann [6], some theoretical properties and applications of EPECs were discussed, namely in engineering sciences and economics. EPEC models are commonly used to study the strategic behavior of firms in deregulated electricity markets. One other important application of EPECs is the multi-leader–follower game in economics, where each leader

E-mail address: wanmei.soon@nie.edu.sg

doi:10.1016/j.amc.2011.06.031

solves a Stackelberg game formulated as an MPEC. As commonly known, an important and frequently considered theoretical issue surrounding such games is the *existence of equilibrium solutions*. Traditionally, such existence results may require certain fixed point theorems (e.g., Kakutani's Fixed Point Theorem and Brouwer fixed point theorem). Generally, it is difficult to ensure the existence of an EPEC solution due to the nonconvexity of the associated MPECs.

Indeed, the theoretical issues facing MPECs alone have appeared in the literature for some time (see Luo et al. [12]). A stochastic version of this problem, namely, a stochastic mathematical program with equilibrium constraints (SMPEC), has also been studied more extensively in recent years. Cromvik and Patriksson investigated the robustness of global optima and stationary solutions to SMPECs in their works [3,4]. The global optima and stationary solutions were shown to be robust with respect to changes in the probability distribution in [3]. Two new extensions of the SMPEC model were proposed in that work. In addition, the authors presented a discretization scheme sample average approximation (SAA) that can be combined with the robustness results to motivate the use of SMPECs in practice. Further in [4], the results in [3] were applied to a classic traffic network design problem (where travel costs are uncertain), and the optimization of a treatment plan in intensity modulated radiation therapy (where the machine parameters and the position of the organs are uncertain). The results from that work point towards great potential in utilizing the SMPEC formalism for modeling and analysis purposes.

The difficulty of asserting the existence and uniqueness of a solution to multi-leader-common-follower games was highlighted in Leyffer and Munson [11], where a characterization of the solution set to such games was presented. Two novel approaches were introduced to solve such games. The authors' numerical results demonstrated that their approaches were practical and competitive with existing methods, in terms of robustness and efficiency. Then in DeMiguel and Xu [5], a stochastic version of a related game was studied, where multiple followers make their decisions after observing the leaders' supply levels and the realized demand function. The existence and uniqueness of the resulting stochastic multiple-leader Stackelberg–Nash–Cournot (SMS) equilibrium was shown. A computational approach to find the equilibrium was proposed based on the SAA method. An application of this framework to model competition in the telecommunication industry was also discussed.

Our work differs from these described above in our focus on NE existence issues specific to our pricing games, which involve MPECs due to the complementarity constraints defining our special demand form. Note that the context surrounding our EPEC is different from those discussed above. In addition, to take into account the uncertainty of the parameters defining demand, a randomized version of the aforementioned game is introduced in this work and its related NE existence issue will be discussed. For clarity, we will first review the pricing game introduced in [17]. A random form of this game is then introduced. In the succeeding section, the case where a single product is offered by each player in the game is investigated. Following that, we go on to discuss the case where multiple products may be offered by each seller. Note that though the first case is a special example of the latter, the NE existence result for the former case can be obtained under weaker conditions. Finally, we determine the existence of NE for a simple random game and conclude with some remarks.

2. A deterministic NCP-constrained pricing model

In [17], a pricing model involving a complementarity-constrained demand system was proposed. We will describe the model here for the convenience of the reader. See the paper therein for details, including the motivation behind our model. Before we discuss the model, we need to present the demand function.

Suppose that in a market there are M sellers and seller i offers N_i distinct products, for each $i \in \{1,\ldots,M\}$. So there are $N_1 + \cdots + N_M = N$ products altogether. We denote the price of product j as p_j and the price vector of all other products as p_{-j}, for each $j \in \{1,\ldots,N\}$. Let $p = (p_1,\ldots,p_N)$. Define D_j as the demand for product j and so $D = (D_1,\ldots,D_N)$ reflects the demand for all the products in the market. As the products can be substitutable for each other or complementary to each other, note that each D_j is a function of p.

Let $d : R^N_+ \to R^N$ be a given function, where $R^N_+ = \{p \in R^N | p \geqslant 0\}$; and $\Omega := \{p \in R^N_+ | d(p) \geqslant 0\}$. For example, if d is of a linear form, it can be written as $d(p) = b - Ap$, where $b \in R^N$ and $A \in R^{N\times N}$. The demand $D : R^N_+ \to R^N_+$ is defined to be d on Ω, i.e. $D(p) = d(p)$ for all $p \in \Omega$. Then for all prices p where $d_j(p) < 0$ for some j, a map B is first used to project them onto Ω. More precisely, B is defined via the complementarity problem:

Definition 1. For any $p \in R^N_+$, $B(p)$ is defined as the solution of the NCP (p): find x (=$B(p)$) such that

$$0 \leqslant d(x) \perp p - x \geqslant 0, \tag{1}$$

where $\perp$ stands for perpendicular and $d(x) \perp p - x \Leftrightarrow d(x)^T(p - x) = 0$.

Note that for all $p \in \Omega$, the projected prices $B(p)$ are exactly p. The demand D is then defined as $D(p) = d(B(p))$ for all $p \in R^N_+$.

As stated in [17], the following assumption is made throughout this work.

Assumption 1. For all $p \in R^N_+$, the NCP (p) has a unique solution $B(p)$ and $B(p) \in \Omega$.