Applied Mathematics and Computation 218 (2011) 1794–1797

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Solitary wave and chaotic behavior of traveling wave solutions for the coupled KdV equations

Jibin Li [a,b,*], Yi Zhang [a]

[a] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, PR China
[b] Kunming University of Science and Technology, Kunming 200062, PR China

ARTICLE INFO

Keywords:
Coupled KdV equation
Traveling wave solution
Chaotic behavior of traveling wave
Solitary wave solution
Periodic wave solution

ABSTRACT

Using the method of dynamical systems to study the coupled KdV system, some exact explicit parametric representations of the solitary wave and periodic wave solutions are obtained in the given parameter regions. Chaotic behavior of traveling wave solutions is determined.

1. Introduction

We consider the following coupled KdV system:

$$A_{1T} + \alpha_1 A_2 A_{1X} + \left(\alpha_2 A_2^2 + \alpha_3 A_1 A_2 + \alpha_4 A_{1XX} + \alpha_5 A_1^2\right)_X = 0, \tag{1a}$$

$$A_{2T} + \delta_1 A_2 A_{1X} + \left(\delta_2 A_1^2 + \delta_3 A_1 A_2 + \delta_4 A_{2XX} + \delta_5 A_2^2\right)_X = 0, \tag{1b}$$

where 10 constants α_i, δ_i ($i = 1,2,3,4,5$) are arbitrary. This system is derived from two-layer fluids, whose integrability and existence of the solitary wave solutions for this system have been discussed by Lou et al. [1]. To our knowledge, the dynamical chaotic behavior of the traveling wave solutions of the corresponding traveling system of (1) have not been considered before. In this paper, by using the method of dynamical systems (see [2]), we further deal with this problem and give possible exact explicit parametric representations of the traveling wave solutions for (1).

To find all traveling wave solutions, letting $\xi = X - cT$, we have

$$-cA_{1\xi} + \alpha_1 A_2 A_{1\xi} + \left(\alpha_2 A_2^2 + \alpha_3 A_1 A_2 + \alpha_4 A_{1\xi\xi} + \alpha_5 A_1^2\right)_\xi = 0, \tag{2a}$$

$$-cA_{2\xi} + \delta_1 A_2 A_{1\xi} + \left(\delta_2 A_1^2 + \delta_3 A_1 A_2 + \delta_4 A_{2\xi\xi} + \delta_5 A_2^2\right)_\xi = 0. \tag{2b}$$

In the following, we always assume that $\alpha_1 = \delta_1 = 0$. For the sake of convenience, we first consider the case $\alpha_4 = \delta_4 = 1$. Integrating two equations of (2) once and taking the integral constants as zeros, we obtain

$$A_1'' = cA_1 - \alpha_2 A_2^2 - \alpha_3 A_1 A_2 - \alpha_5 A_1^2, \tag{3a}$$

$$A_2'' = cA_2 - \delta_2 A_1^2 - \delta_3 A_1 A_2 - \delta_5 A_2^2. \tag{3b}$$

This is a 8-parameter system, which has very complicated dynamical behavior, and the complete study for (3) is more difficult.

* Corresponding author at: Department of Mathematics, Zhejiang Normal University, Jinhua 321004, PR China.
E-mail addresses: lijb@zjnu.cn (J. Li), zy2836@163.com (Y. Zhang).

0096-3003/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2011.06.063

Suppose that $\alpha_2 = \frac{1}{2}\delta_3,\ \alpha_3 = 2\delta_2$, and let $q_1 = A_1,\ q_2 = A_2,\ p_1 = A_1',\ p_2 = A_2'$. Then, we have the following Hamiltonian system with two degree of freedom:

$$\frac{d}{d\xi}\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -cq_1 + \alpha_2 q_2^2 + 2\delta_2 q_1 q_2 + \alpha_5 q_1^2 \\ -cq_2 + \delta_2 q_1^2 + 2\alpha_2 q_1 q_2 + \delta_5 q_2^2 \\ p_1 \\ p_2 \end{pmatrix} = J\nabla H, \tag{4}$$

where the Hamiltonian H is given by

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1^2 - cq_1^2) + \frac{1}{2}(p_2^2 - cq_2^2) + \frac{1}{3}\alpha_5 q_1^3 + \frac{1}{3}\delta_5 q_2^3 + \delta_2 q_1^2 q_2 + \alpha_2 q_2^2 q_1. \tag{5}$$

Based on the method of dynamical systems, we consider system (3) and system (4), respectively, in next two sections.

2. The exact explicit solitary wave solution and periodic wave solutions determined by (3)

In this section, we consider the subspace $A_1 = \omega A_2$ in the four-dimensional phase space (A_1, A_2, A_1', A_2'). Substituting $A_1 = \omega A_2$ into (3), we have

$$A_2'' = cA_2 - \left(\frac{\alpha_2}{\omega} + \alpha_3 + \omega\alpha_5\right)A_2^2, \tag{6a}$$

$$A_2'' = cA_2 - (\omega^2\delta_2 + \omega\delta_3 + \delta_5)A_2^2. \tag{6b}$$

Clearly, if and only if $\frac{\alpha_2}{\omega} + \alpha_3 + \omega\alpha_5 = \omega^2\delta_2 + \omega\delta_3 + \delta_5$, i.e., ω is a real root of the cubic algebraic equation

$$\delta_2\omega^3 + (\delta_3 - \alpha_5)\omega^2 + (\delta_5 - \alpha_3)\omega - \alpha_2 = 0, \tag{7}$$

then, (6a) and (6b) define the same solution. We next assume that ω satisfies (7) and denote that $B = \omega^2\delta_2 + \omega\delta_3 + \delta_5$. Eq. (6b) can be written as the following planar system

$$\frac{dA_2}{d\xi} = y, \quad \frac{dy}{d\xi} = cA_2 - (\omega^2\delta_2 + \omega\delta_3 + \delta_5)A_2^2 \equiv cA_2 - BA_2^2, \tag{8}$$

which has the Hamiltonian

$$H_2(A_2, y) = \frac{1}{2}y^2 - \frac{1}{2}cA_2^2 + \frac{1}{3}BA_2^3. \tag{9}$$

System (8) has two equilibrium points $O(0,0)$ and $E(A_{20},0)$, where $A_{20} = \frac{c}{B}$. It is easy to see that when $c > 0$ (<0), $O(0,0)$ is a saddle point (a center); $E(A_{20},0)$ is a center (a saddle point). Notice that $h_0 = H_2(0,0) = 0,\ h_1 = H_2(A_{20},0) = -\frac{c^3}{6B^2}$. Thus, we obtain the following conclusion.

Theorem 1. *Suppose that $\alpha_1 = \delta_1 = 0$, $B = \omega^2\delta_2 + \omega\delta_3 + \delta_5$ and ω is a real root of (7).*

(1) When $c > 0$, $B > 0$, (9) can be written as $y^2 = \frac{2}{3}B\left(\frac{3h}{B} + \frac{3c}{2B}A_2^2 - A_2^3\right) = \frac{2}{3}B(a - A_2)(A_2 - b)(A_2 - e), \quad h \in (h_1, h_0)$.

(i) Corresponding to the family of periodic orbits of (8) defined by $H_2(A_2, y) = h$, $h \in (h_1, h_0)$, Eq. (1) has a family of periodic wave solutions, which has the following parametric representation

$$A_2(\xi) = a - (a - b)\mathrm{sn}^2(\Omega_0\xi, k), \tag{10}$$

where $\Omega_0 = \sqrt{\frac{B(a-e)}{6}}, \quad k = \sqrt{\frac{a-b}{a-e}}$ and $A_1(\xi) = \omega A_2(\xi)$.

(ii) Corresponding to the homoclinic orbit of (8) defined by $H_2(A_2, y) = 0$, Eq. (1) has a solitary wave solutions of peak type, which has the parametric representation

$$A_2(\xi) = \frac{3c}{2B}\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}\xi\right), \quad A_1(\xi) = \omega A_2(\xi). \tag{11}$$

(2) When $c < 0$, $B > 0$, the origin $O(0,0)$ becomes a center, while E is a saddle point. (9) can be written as $y^2 = \frac{2}{3}B\left(\frac{3h}{B} + \frac{3c}{2B}A_2^2 - A_2^3\right) = \frac{2}{3}B(a_1 - A_2)(A_2 - b_1)(A_2 - e_1), \quad h \in (0, h_1)$.

(i) Corresponding to the family of periodic orbits of (8) defined by $H_2(A_2, y) = h$, $h \in (0, h_1)$, Eq. (1) has a family of periodic wave solutions, which has the parametric representation

$$A_2(\xi) = a_1 - (a_1 - b_1)\mathrm{sn}^2(\Omega_1\xi, k_1), \tag{12}$$