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Split-step orthogonal spline collocation methods for nonlinear Schrödinger equations in one, two, and three dimensions

Shanshan Wang, Luming Zhang *

College of Science, Nanjing University of Aeronautics and Astronautics, 29 Yudao Street, Nanjing 210016, China

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ABSTRACT

Split-step orthogonal spline collocation (OSC) methods are proposed for one-, two-, and three-dimensional nonlinear Schrödinger (NLS) equations with time-dependent potentials. Firstly, the NLS equation is split into two nonlinear equations, and one or more one-dimensional linear equations. Commonly, the nonlinear subproblems could be integrated directly and accurately, but it fails when the time-dependent potential cannot be integrated exactly. In this case, we propose three approximations by using quadrature formulae, but the split order is not reduced. Discrete-time OSC schemes are applied for the linear subproblems. In numerical experiments, many tests are carried out to prove the reliability and efficiency of the split-step OSC (SSOSC) methods. Solitons in one, two, and three dimensions are well simulated, and conservative properties and convergence rates are demonstrated. We also apply the ways of solving the nonlinear subproblems to the split-step finite difference (SSFD) methods and the time-splitting spectral (TSSP) methods, and the approximate ways still work well. Finally, we apply the SSOSC methods to solve some problems of Bose–Einstein condensates.

1. Introduction

In this paper, we consider the following initial-boundary value problem (IBVP) of the nonlinear Schrödinger (NLS) equation:

$$i\frac{\partial\psi(\mathbf{x},t)}{\partial t}+\alpha\nabla^2\psi(\mathbf{x},t)+V_d(\mathbf{x},t)\psi(\mathbf{x},t)+\beta_d|\psi(\mathbf{x},t)|^2\psi(\mathbf{x},t)=0,\quad \mathbf{x}\in\Omega_d,\ t\in[0,T], \tag{1}$$

$$\psi(\mathbf{x},0)=\psi_0(\mathbf{x}),\quad \mathbf{x}\in\Omega_d, \tag{2}$$

$$\psi(\mathbf{x},t)=0,\quad \mathbf{x}\in\partial\Omega_d,\ t\in[0,T], \tag{3}$$

where $\Omega_d\in\mathbb{R}^d$ $(d=1,2,3)$, $\partial\Omega_d$ is the boundary of Ω_d, α and β are two real constants, and $i=\sqrt{-1}$. $V_d(\mathbf{x},t)$ is a given real function. When $V_d(\mathbf{x},t)\equiv 0$, Eq. (1) becomes the usual cubic NLS equation [1]. When $\alpha = 1/2$ and

$$V_d(\mathbf{x},t)\equiv V_d(\mathbf{x})=\begin{cases}-\frac{1}{2}x^2, & d=1,\\ -\frac{1}{2}\left(x^2+\gamma_y^2y^2\right), & d=2,\\ -\frac{1}{2}\left(x^2+\gamma_y^2y^2+\gamma_z^2z^2\right), & d=3,\end{cases} \tag{4}$$

* Corresponding author.
E-mail addresses: wangss@nuaa.edu.cn (S. Wang), zhanglm@nuaa.edu.cn (L. Zhang).

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Eq. (1) is the Gross–Pitaevskii equation studied in [2,3], which is usually used to model the properties of a Bose–Einstein condensate (BEC) at extremely low temperatures.

Computing the inner product of Eq. (1) with ψ, and taking the imaginary part, we have

$$Q(t) = \int_{\Omega_d} |\psi(\mathbf{x},t)|^2 d\mathbf{x} = \int_{\Omega_d} |\psi(\mathbf{x},0)|^2 d\mathbf{x} = Q(0). \tag{5}$$

This is a conserved quantity of the IBVP (1)–(3). When $V_d(\mathbf{x},t) \equiv V_d(\mathbf{x})$, we compute the inner product of Eq. (1) with $\partial\psi/\partial t$, and take the real part of the result. Then we obtain another conserved quantity as follows:

$$E(t) = \int_{\Omega_d} \left[\alpha|\nabla\psi(\mathbf{x},t)|^2 - V_d(\mathbf{x})|\psi(\mathbf{x},t)|^2 - \frac{\beta_d}{2}|\psi(\mathbf{x},t)|^4\right] d\mathbf{x} = E(0). \tag{6}$$

Many studies have been made for NLS equations, and one may refer to Refs. [2–5] and references therein. In these research works, the split step technique is interesting which could be combined with the spectral method [2,4] or the finite difference (FD) method [3]. In this paper, we try to apply this technique with discrete-time orthogonal spline collocation (OSC) methods for NLS equations in one, two, and three dimensions, respectively.

Several researches on the OSC method for NLS equations have been made. For one dimension, semi-discretization OSC method is utilized for the cubic NLS equation [6], and is extended to the equation with power nonlinearity and also to the generalized one [7]. In [8,9], alternating direction implicit (ADI) OSC method is analyzed for the linear Schrödinger equation in two space variables. ADI OSC method is also applied to the two-dimensional NLS equation [9], where extrapolated technique is used for the nonlinear term. So far, few OSC methods have been found to consider three-dimensional problems.

In this paper, the OSC approach is combined with the split-step method to construct new schemes named after split-step OSC (SSOSC) schemes. These schemes are effective for solving the one-, two-, and also for the three-dimensional NLS equations. Especially for the multidimensional cases, the SSOSC methods could be implemented smoothly by utilizing the fundamental property of the piecewise Hermite interpolation. And owing to the split step technique, the nonlinear term is dealt with easily without the extrapolation.

This paper is organized as follows. In Section 2, we introduce some preliminaries. Section 3 is devoted to formulate split-step OSC methods. In Section 4, the implementation of the SSOSC schemes are discussed. Extensive experiments are carried out in Section 5. Finally, Section 6 draws some conclusions.

2. Preliminaries

Let $\Omega_3 = [x_L,x_R] \times [y_L,y_R] \times [z_L,z_R]$, and $\{x_k\}_{k=0}^{N_x}$, $\{y_l\}_{l=0}^{N_y}$ and $\{z_m\}_{m=0}^{N_z}$ be partitions of $[x_L,x_R]$, $[y_L,y_R]$ and $[z_L,z_R]$, respectively, such that

$$\begin{aligned}
&x_L = x_0 < x_1 < \cdots < x_{N_x-1} < x_{N_x} = x_R,\\
&y_L = y_0 < y_1 < \cdots < y_{N_y-1} < y_{N_y} = y_R,\\
&z_L = z_0 < z_1 < \cdots < z_{N_z-1} < z_{N_z} = z_R.
\end{aligned}$$

Denote $h_k^x = x_k - x_{k-1},\ k = 1,2,\ldots,N_x,\ h_l^y = y_l - y_{l-1},\ l = 1,2,\ldots,N_y$ and $h_m^z = z_m - z_{m-1},\ m = 1,2,\ldots,N_z$. We divide the interval $[0,T]$ by the partition $\{t_n\}_{n=0}^{J}$, where $t_n = n\tau$ and $\tau = T/J$.

Let $\mathcal{M}_x^0,\ \mathcal{M}_y^0$ and $\mathcal{M}_y^0$ be spaces of piecewise Hermite cubic defined by

$$\begin{aligned}
&\mathcal{M}_x^0 = \{v \in C^1[x_L,x_R] : v|_{[x_{k-1},x_k]} \in P_3, 1 \leqslant k \leqslant N_x\} \cap \{v(x_L) = v(x_R) = 0\},\\
&\mathcal{M}_y^0 = \{v \in C^1[y_L,y_R] : v|_{[y_{l-1},y_l]} \in P_3, 1 \leqslant l \leqslant N_y\} \cap \{v(y_L) = v(y_R) = 0\},\\
&\mathcal{M}_z^0 = \{v \in C^1[z_L,z_R] : v|_{[z_{m-1},z_m]} \in P_3, 1 \leqslant m \leqslant N_z\} \cap \{v(z_L) = v(z_R) = 0\},
\end{aligned}$$

where P_3 denotes the set of all polynomials of degree $\leqslant 3$. Let $\mathcal{M}_{xy}^0 = \mathcal{M}_x^0 \otimes \mathcal{M}_y^0$ be the space of piecewise Hermite bicubic, and $\mathcal{M}_{xyz}^0 = \mathcal{M}_{xy}^0 \otimes \mathcal{M}_z^0$ be the space of piecewise Hermite polynomials in x, y and z.

Let $\mathcal{G}_x = \left\{\xi_{k,k_1}^x\right\}_{k,k_1=1}^{N_x,2}$, $\mathcal{G}_y = \left\{\xi_{l,l_1}^y\right\}_{l,l_1=1}^{N_y,2}$ and $\mathcal{G}_z = \left\{\xi_{m,m_1}^z\right\}_{m,m_1=1}^{N_z,2}$ be the sets of Gauss points defined by

$$\begin{aligned}
&\xi_{k,k_1}^x = x_{k-1} + h_k^x\lambda_{k_1}, \quad k = 1,2,\ldots,N_x,\ k_1 = 1,2,\\
&\xi_{l,l_1}^y = y_{l-1} + h_l^y\lambda_{l_1}, \quad l = 1,2,\ldots,N_y,\ l_1 = 1,2,\\
&\xi_{m,m_1}^z = z_{m-1} + h_m^z\lambda_{m_1}, \quad m = 1,2,\ldots,N_z,\ m_1 = 1,2,
\end{aligned}$$

where

$$\lambda_1 = (3 - \sqrt{3})/6, \quad \lambda_2 = (3 + \sqrt{3})/6,$$

which are nodes of the 2-point Gauss–Legendre quadrature on $[0,1]$ with corresponding weights $\omega_1 = \omega_2 = 1/2$. Denote