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Numerical methods for Fredholm integral equations on the square

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ABSTRACT

In this paper we shall investigate the numerical solution of two-dimensional Fredholm integral equations by Nyström and collocation methods based on the zeros of Jacobi orthogonal polynomials. The convergence, stability and well conditioning of the method are proved in suitable weighted spaces of functions. Some numerical examples illustrate the efficiency of the methods.

1. Introduction

This paper deals with the numerical approximation of the solution of Fredholm integral equations of the second kind, defined on the square $S = [-1,1]^2$,

$$f(x,y) - \mu \int_S k(x,y,s,t) f(s,t) w(s,t)\, ds\, dt = g(x,y), \quad (x,y) \in S, \tag{1.1}$$

where $w(x,y) := v^{\alpha_1,\beta_1}(x) v^{\alpha_2,\beta_2}(y) = (1-x)^{\alpha_1}(1+x)^{\beta_1}(1-y)^{\alpha_2}(1+y)^{\beta_2}, \alpha_1, \beta_1, \alpha_2, \beta_2 > -1, \mu \in \mathbb{R}$. k and g are given functions defined on $[-1,1]^4$ and $[-1,1]^2$, respectively, which are sufficiently smooth on the open sets but can have (algebraic) singularities on the boundaries. f is the unknown function.

This topic is of interest, since many problems in different areas, like computer graphics, engineering, mathematical physics, etc., can be modeled by bivariate Fredholm equations of the second kind (see for instance the rendering equation in [9,8]). Some of the existing numerical procedures make use of collocation or Nyström methods based on piecewise approximating polynomials [2,10] or Monte Carlo methods [8], or discrete Galerkin methods [7]. In this paper, following a well known approach in the one dimensional case (see for instance [3] and the reference therein), we propose a global approximation of the solution by means of a Nyström method based on a cubature rule obtained as the tensor product of two univariate Gaussian rules and a polynomial collocation method, both based on Jacobi zeros. The reasons why this approach is not trivial is that there are very few results in the literature about the polynomial approximation in two variables.

Moreover the additional difficulty of considering functions which can have singularities on the boundaries can be treated only by introducing weighted approximation schemes and weighted spaces of functions (see [5] for the one dimensional case).

Therefore some preliminary results about the best polynomial approximation, the Lagrange interpolation based on Jacobi zeros and the tensor product of Gaussian rules are given in weighted spaces of functions.

As it is well-known, in both the proposed methods we have to solve a system of linear equations. Here, under suitable conditions, we prove that the system is uniquely solvable and well-conditioned. Moreover we prove that both methods are stable and convergent, giving error estimates in suitable Sobolev spaces, equipped with the weighted uniform norm.

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Finally, as an application, we take under consideration the case of integral equations, defined on different domains, reducible by suitable transformations to equation of the type (1.1). In this sense the hypothesis of the squared domain is not a restriction. However, as we will see, the price to be payed is that the smoothness of the involved functions can be lost by the change of variables (see Example 4).

The outline of this paper is as follows. Section 2 contains the notation and announced auxiliary results about the bivariate Lagrange interpolation and the cubature rule. The main results are given in the Section 3. Section 4 contains the computational details about the construction of the approximating solutions and some numerical tests that show the performance of our procedures and confirm the theoretical results. Section 5 contains the proofs of the main results. Finally Appendix is devoted to the estimate of the bivariate best polynomial approximation error in weighted L_p norm, $1 < p \leqslant \infty$, in terms of the best polynomial approximation w.r.t. the single variables.

2. Notations and preliminary results

In this section we define the spaces of functions in which we will study the equations under consideration. Moreover we will give some basic tools of the polynomial approximation theory in two variables.

Since we are considering the case of functions which can have singularities on the boundaries of the square S, we introduce a weighted space of functions as follows: the weight $u(x,y) = v^{\gamma_1,\delta_1}(x)v^{\gamma_2,\delta_2}(y),\ \gamma_1,\delta_1,\gamma_2,\delta_2 \geqslant 0$, fixed, we define the space C_u,

$$C_u = \left\{ f \in C((-1,1)^2) : \begin{array}{ll} -1 \leqslant y \leqslant 1, & \lim\limits_{x\to\pm 1} f(x,y)u(x,y) = 0, \\ -1 \leqslant x \leqslant 1, & \lim\limits_{y\to\pm 1,} f(x,y)u(x,y) = 0, \end{array} \right\}$$

where the limit conditions hold uniformly w.r.t. the free variable. Whenever one or more of the parameters γ_1, δ_1, γ_2, δ_2 is greater than 0, the functions in C_u can be singular on one or more sides of the square S. In the case $\gamma_1 = \delta_1 = \gamma_2 = \delta_2 = 0$ the definition reduces to the case of continuous functions and we set $C_u = C(S)$.

C_u can be equipped with the weighted uniform norm on the square

$$\|f\|_{C_u} = \|fu\|_\infty = \sup_{(x,y)\in S} |f(x,y)u(x,y)|.$$

Now set $\varphi_1(x) = \sqrt{1-x^2},\ \varphi_2(y) = \sqrt{1-y^2}$ and denote by f_x and f_y the function $f(x,y)$ as a function of the only variable y or x respectively. For smoother functions, i.e. for functions having some derivatives which can be discontinuous on the boundaries of S, we introduce the following Sobolev-type space

$$W_r(u) = \left\{ f \in C_u : M_r(f,u) := \sup\left\{ \left\| f_y^{(r)}\varphi_1^r u \right\|_\infty, \left\| f_x^{(r)}\varphi_2^r u \right\|_\infty \right\} < \infty \right\}, \tag{2.2}$$

where the superscript (r) denotes the rth derivative of the one-dimensional function f_y or f_x. We equip $W_r(u)$ with the norm

$$\|f\|_{W_r(u)} = \|fu\|_\infty + M_r(f,u).$$

In the case $u(x,y) \equiv 1$ we will simply write W_r.

Now let $\mathbb{P}_{m,m}$ denote the space of all algebraic polynomials of two variables of degree at most m in each variable.

The error of best polynomial approximation in C_u by means of bivariate polynomials in $\mathbb{P}_{m,m}$ can be defined as follows

$$E_{m,m}(f)_u = \inf_{P\in\mathbb{P}_{m,m}} \|[f-P]u\|_\infty.$$

Now if h is a continuous function on $(-1,1)$ and $v^{\gamma,\delta}(x) = (1-x)^\gamma(1+x)^\delta$, γ, $\delta \geqslant 0$, let $E_m(h)_{v^{\gamma,\delta}} := \inf_{p\in\mathbb{P}_m} \|[h-p]v^{\gamma,\delta}\|_\infty$ be the weighted error of best approximation of the univariate function h, where $\mathbb{P}_m$ denotes the set of the univariate polynomials of degree at most m.

Using the definitions given above it is possible to prove that (a general result, including the L^p case is stated and proved in Appendix)

$$E_{m,m}(f)_u \leqslant \mathcal{C}\left[\sup_{x\in[-1,1]} v^{\gamma_1,\delta_1}(x) E_{\left[\frac{m+1}{2}\right]}(f_x)_{v^{\gamma_2,\delta_2}} + \sup_{y\in[-1,1]} v^{\gamma_2,\delta_2}(y) E_{\left[\frac{m+1}{2}\right]}(f_y)_{v^{\gamma_1,\delta_1}} \right], \tag{2.3}$$

where the positive constant $\mathcal{C}$ does not depend on f and m. In other words, the bivariate best approximation error can be evaluated by means of the univariate best approximation errors w.r.t. each of the two variables of f. Therefore using this estimate and the Favard inequality, it follows that if $f \in W_r(u)$ then

$$E_{m,m}(f)_u \leqslant \mathcal{C}\frac{M_r(f,u)}{m^r}, \tag{2.4}$$

where the positive constant $\mathcal{C}$ does not depend on f and m.

Along all the paper the constant $\mathcal{C}$ will be used several times, having different meaning in different formulas. Moreover from now on we will write $\mathcal{C} \neq \mathcal{C}(a,b,\ldots)$ in order to say that $\mathcal{C}$ is a positive constant independent of the parameters $a,b,\ldots,$