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Power series solutions to Volterra integral equations

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ABSTRACT

Power series type solutions are given for a wide class of linear and q-dimensional nonlinear Volterra equations on R^p. The basic assumption on the kernel $K(\mathbf{x},\mathbf{y})$ is that $K(\mathbf{x},\mathbf{xt})$ has a power series in $\mathbf{x}$. For example, this holds for any analytic kernel.

The kernel may be strongly singular, provided certain constants are finite. One and only one such power series solution exists. Its coefficients are given by a simple iterative formula. In many cases this may be solved explicitly. In particular an explicit formula for the resolvent is given.

1. Introduction

Volterra integral equations are a special type of integral equations introduced by Vito Volterra. They have applications in demography, the study of viscoelastic materials, and in insurance mathematics through the renewal equation. There has been a great deal of developments on the theory and applications of Volterra integral equations. For most comprehensive accounts, see Agarwal et al. [2] and Agarwal and O'Regan [1].

This note shows that a very large class of Volterra integral equations on R^p have easily computed power series solutions. We refer here not to the Neumann power series in λ but to powers of the "$\mathbf{x}$" variable in R^p. This method of solution is appealing because it is straightforward and simple to use, and once found at a point $\mathbf{x} > \mathbf{0}$ in R^p is also found throughout $\mathbf{0} \leqslant \mathbf{y} \leqslant \mathbf{x}$. So, it avoids any need for numerical integration. We have not been able to find any references to this method in Mathematical Reviews or any text on integral equations.

We assume that the kernel can be expressed in the form

$$K(\mathbf{x},\mathbf{xt}) = \sum_{\mathbf{n}=\mathbf{I}-\mathbf{1}}^{\infty} \mathbf{x}^{\mathbf{n}} k_{\mathbf{n}}(\mathbf{t}), \qquad (1.1)$$

for $\mathbf{x}$ and $\mathbf{t}$ in R^p, where $\mathbf{I}$ is a (vector) integer in R^p, $\mathbf{1}$ is the vector of 1's, summation is over integers $\mathbf{n} \geqslant \mathbf{I}-\mathbf{1}$ in R^p, that is over $n_1 \geqslant I_1 - 1, \ldots, n_p \geqslant I_p - 1$,

$$\mathbf{xt} = (x_1 t_1, \ldots, x_p t_p), \quad \mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_p^{n_p}.$$

When (1.1) involves only a finite sum, the kernel is not degenerate in the usual sense. For example, the kernel

$$K(\mathbf{x},\mathbf{y}) = \mathbf{x}^{\mathbf{a}} (\mathbf{x}-\mathbf{y})^{\mathbf{b}} \mathbf{y}^{\mathbf{c}}, \qquad (1.2)$$

where $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{m}$, an integer, satisfies (1.1) with only one term, $\mathbf{x}^{\mathbf{m}} k_{\mathbf{m}}(\mathbf{t})$, where $k_{\mathbf{m}}(\mathbf{t}) = (\mathbf{1}-\mathbf{t})^{\mathbf{b}} \mathbf{t}^{\mathbf{c}}$. Its resolvent is given explicitly in Section 6.

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Sections 2 and 3 deal with linear equations of the second and first kind. Sections 4 and 5 deal with *nonlinear* equations of the second and first kind.

In each case, the power series type solution is found to be unique – even if the kernel is strongly singular. The nth coefficient of the power series solution is given in terms of the previous coefficients. There is no need to deal with equations of the first kind by transforming them to equations of the second kind. In many cases the iterative solution may be solved explicitly. In particular this may be done for linear equations of the second kind: an explicit form for their resolvent is given in Section 6.

Section 7 considers what kernels are transformable to type (1.1) by one to one transformations. For example, it shows that this allows us to drop the requirement in (1.2) that $\mathbf{a}+\mathbf{b}+\mathbf{c}$ be an integer.

If $\mathbf{I}$ is not positive there is no guarantee that the solution found is unique. Indeed in Section 8 we obtain nontrivial solutions to $f(\mathbf{x}) = \int_{\mathbf{0}}^{\mathbf{x}} K(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y}$ when $\mathbf{I}=\mathbf{0}$ by an extension of Frobenius's technique.

2. Linear equations of the second kind

In this section, we give an iterative solution to

$$f(\mathbf{x}) = g(\mathbf{x}) + \int_{\mathbf{0}}^{\mathbf{x}} K(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y} \tag{2.1}$$

on $\Omega = R^p$ or $(0,\infty)^p$. A different form of this solution will be given in Section 6.

In order to allow for 'forcing functions' $g(\mathbf{x})$ such as $\mathbf{x}^{1/2}+\mathbf{x}^{-1/3}$ as well as analytic functions, we shall assume $g(\mathbf{x})$ can be expressed in the form

$$g(\mathbf{x}) = \sum_{\mathbf{n}=\mathbf{L}}^{\infty} \mathbf{x}^{\mathbf{n}} \int_{-\mathbf{1}}^{\mathbf{0}} \mathbf{x}_+^{\mathbf{u}} g_{\mathbf{n}}(\mathbf{u})d\nu(\mathbf{u}), \tag{2.2}$$

where $\mathbf{L}$ is a given integer in R^p, $\mathbf{x}_+$ is the vector in R^p with ith component $|x_i|$, ν is a measure on $(-\mathbf{1},\mathbf{0}] \subset R^p$, and $-\mathbf{1}$ is the vector $(-1,\ldots,-1)'$.

So, essentially we are dealing with $g(\mathbf{x})$ of the form $\int_{\mathbf{L}-\mathbf{1}}^{\infty} \mathbf{x}^{\mathbf{u}} d\bar{\nu}(\mathbf{u})$. For example, $g(\exp(-\mathbf{x}))$ may be any Laplace transform on R^p, where $\exp(-\mathbf{x}) = (\exp(-x_1),\ldots,\exp(-x_p))$.

It is easy to see that (2.1) has a solution of the same form

$$f(\mathbf{x}) = \sum_{\mathbf{n}=\mathbf{L}}^{\infty} \mathbf{x}^{\mathbf{n}} \int_{\mathbf{1}}^{\mathbf{0}} \mathbf{x}_+^{\mathbf{u}} f_{\mathbf{n}}(\mathbf{u})d\nu(\mathbf{u}), \tag{2.3}$$

if and only if for $\mathbf{u}$ in $(-\mathbf{1},\mathbf{0}]^p$ any point of change of ν,

$$(f_n(\mathbf{u}) - g_n(\mathbf{u}))\mathbf{I}(\mathbf{n} \geqslant \mathbf{L}) = b_{\mathbf{n}}(\mathbf{u})I(\mathbf{n} \geqslant \mathbf{I}+\mathbf{L}), \tag{2.4}$$

where

$$b_{\mathbf{n}}(\mathbf{u}) = \sum_{\mathbf{m}=\mathbf{L}}^{\mathbf{n}-\mathbf{I}} a_{\mathbf{nm}}(\mathbf{u})f_{\mathbf{m}}(\mathbf{u}), a_{\mathbf{nm}}(\mathbf{u}) = \int_{\mathbf{0}}^{\mathbf{1}} \mathbf{t}^{\mathbf{m}+\mathbf{u}} k_{\mathbf{n}-\mathbf{m}-\mathbf{1}}(\mathbf{t})d\mathbf{t}.$$

We now drop the argument $\mathbf{u}$ and show how to solve (2.4) interactively.

There are three overlapping cases, depending on whether $\mathbf{I}$ has positive or negative components or both. Set $\mathbf{I}_+ = \max(\mathbf{I},\mathbf{0})$ and $\mathbf{I}_- = \max(-\mathbf{I},\mathbf{0})$ componentwise. So, $\mathbf{I} = \mathbf{I}_+ - \mathbf{I}_-$.

Case 1: $\mathbf{I} \not\leqslant \mathbf{0}$, that is $\mathbf{I}_+ \neq \mathbf{0}$, that is $\mathbf{I}$ has at least one component positive. An iterative solution is given by

$$f_{\mathbf{n}} = g_{\mathbf{n}} + \sum_{\mathbf{m}=\mathbf{L}}^{\mathbf{n}-\mathbf{I}} a_{\mathbf{nm}} f_{\mathbf{m}}$$

for $\mathbf{n} \geqslant \mathbf{L}$, assuming that these $\{a_{\mathbf{nm}}\}$ are finite. (The argument $\mathbf{u}$ has been suppressed. A sum is taken to be zero if its range is empty: in this case, if $\mathbf{n} \not\geqslant \mathbf{I}+\mathbf{L}$.) For example,

$$f_{\mathbf{n}} = \begin{cases} g_{\mathbf{n}}, & \text{for } \mathbf{L} \leqslant \mathbf{n} \geqslant \mathbf{L}+\mathbf{I}, \\ g_{\mathbf{n}} + \sum\limits_{\mathbf{m}=\mathbf{L}}^{\mathbf{n}-\mathbf{I}} a_{\mathbf{nm}} g_{\mathbf{m}} = G_{\mathbf{n}} \text{ say}, & \text{for } \mathbf{L}+\mathbf{I}_+ \leqslant \mathbf{n} \not\geqslant \mathbf{L}+2\mathbf{I}, \\ g_{\mathbf{n}} + \sum\limits_1 a_{\mathbf{nm}} g_{\mathbf{m}} + \sum\limits_2 a_{\mathbf{nm}} G_{\mathbf{m}}, & \text{for } \mathbf{L}+2\mathbf{I}_+ \leqslant \mathbf{n} \not\geqslant \mathbf{L}+3\mathbf{I}, \end{cases}$$

where $\sum_1$ sums over $\{\mathbf{L} \leqslant \mathbf{m} \not\geqslant \mathbf{L}+\mathbf{I}\}$ and $\sum_2$ over $\{\mathbf{L}+\mathbf{I}_+ \leqslant \mathbf{m} \leqslant \mathbf{n}-\mathbf{I}\}$.

Case 2: $\mathbf{I} \not\geqslant \mathbf{0}$, that is $\mathbf{I}_- \neq \mathbf{0}$, that is $\mathbf{I}$ has at least one component negative. An iterative solution is given by

$$f_{\mathbf{n}} = \begin{cases} 0, & \text{if } \mathbf{L} \leqslant \mathbf{n} \not\geqslant \mathbf{L}-\mathbf{I}, \\ \bar{a}_{\mathbf{nn}}^{-1}\left(-g_{\mathbf{n}+\mathbf{I}} + f_{\mathbf{n}+\mathbf{I}} - \sum\limits_{\mathbf{m}=\mathbf{L}}^{\prime\mathbf{n}} \bar{a}_{\mathbf{nm}} f_{\mathbf{m}}\right), & \text{for } \mathbf{n} \geqslant \mathbf{L}+\mathbf{I}_-, \end{cases}$$