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On convergence of a new secant-like method for solving nonlinear equations

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ABSTRACT

In this paper, we prove that the order of a new secant-like method presented recently with the so-called order of 2.618 is only 2.414. Some mistakes in the derivation leading to such a conclusion are pointed out. Meanwhile, under the assumption that the second derivative of the involved function is bounded, the convergence radius of the secant-like method is given, and error estimates matching its convergence order are also provided by using a generalized Fibonacci sequence.

1. Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. In this study, we apply iterative methods to find a simple root $x^\star$ of a nonlinear equation $f(x) = 0$, where $f: D \subset \mathbf{R} \to \mathbf{R}$ is a scalar function on an open interval D.

Newton's method is the well-known iterative method to find $x^\star$ by using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n \geqslant 0)\ (x_0 \in D), \tag{1}$$

which converges quadratically in some neighborhood of $x^\star$ (see [1,2]). But Newton's method requires evaluations of first order derivatives at each step. To avoid this, we can use the secant method (see [1]) instead, as follows:

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n), \quad (n \geqslant 0)\ (x_0, x_{-1} \in D). \tag{2}$$

The main limitation of this method with respect to Newton's method is the order. It is Q-superlinearly convergent with order $p = \frac{1+\sqrt{5}}{2} = 1.618\ldots$

Recently, Zhang et al. in Ref. [3] proposed a new two-step secant-like method as follows:

$$\begin{cases} x_{n+1} = x_n - \frac{f(x_n)}{f(y_n) - f(x_n)}(y_n - x_n) \\ y_{n+1} = x_{n+1} - \frac{f(x_{n+1})}{f(y_n) - f(x_n)}(y_n - x_n) \end{cases} \quad (n \geqslant 0)\ (x_0, y_0 \in D). \tag{3}$$

They proved that the convergence order of method (3) is 2.618.

The first impression is that the above conclusion on the order of method (3) may be wrong. Since method (3) is very similar to the well-known King–Werner's method with order of 2.414, which has the form [4,5]

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$$\begin{cases} x_{n+1} = x_n - f'\left(\frac{x_n+y_n}{2}\right)^{-1} f(x_n), \\ y_{n+1} = x_{n+1} - f'\left(\frac{x_n+y_n}{2}\right)^{-1} f(x_{n+1}), \end{cases} (n \geqslant 0)\ (x_0, y_0 \in D). \tag{4}$$

In fact, we can obtain iteration (3) from King–Werner's method (4) by simply replacing the first order derivatives by the first order divided differences. Generally speaking, such a replacement cannot improve the order of an iteration. On the contrary, this replacement may reduce the order. For example, the order of the secant method (2) reduces from 2 to 1.618, just due to the replacement of the first order derivatives in Newton's method (1) by the first order divided differences.

In this paper, we prove that the convergence order of the secant-like method (3) is only 2.414. Some mistakes in the proof of the main theorem of Zhang et al.'s paper are pointed out.

It is known that, the classical local convergence theorem for an iterative method only shows that the iterative sequence generated by this method is well defined, and converges to a solution $x^\star$ of the involved function, provided that the initial points are sufficiently close to the solution $x^\star$. Note that Zhang et al.'s paper established this type of local convergence theorem for method (3).

The main limitation of the above classical local theorem is that it tells us no information on the convergence radius of an iterative method. Here, we say the convergence radius of an iterative method is r, which means that there is an open ball $B(x_\star, r) \subseteq D$ with center $x_\star$ and radius r, so that the sequence generated by this iterative method starting from any initial points in it converges to $x^\star$. Of course, we will be delighted to have bigger convergence radius of an iterative method, but this depends not only on the iterative method we use but also on the conditions of the involved function, such as Lipschitz continuous conditions, Hölder continuous conditions, etc.

Traub and Wozniakowski in [6] and Wang in [7] gave an exact estimate of convergence radius $r = \frac{2}{3K}$ respectively for Newton's method (1) under the following Lipschitz continuous condition

$$\left|f'(x_\star)^{-1}(f'(x) - f'(y))\right| \leqslant K|x - y|, \quad \forall x, y \in D, \quad \text{for some } K > 0. \tag{5}$$

Using a combination of (5) and the center-Lipschitz condition

$$\left|f'(x_\star)^{-1}(f'(x) - f'(x_\star))\right| \leqslant K_0|x - x_\star|, \quad \forall x \in D, \quad \text{for some } K_0 > 0. \tag{6}$$

Argyros [8] provided the radius $\frac{2}{2K_0+K}$ which enlarges $\frac{2}{3K}$ for $K_0 < K$ under the same computational cost. Huang in [9] generalized Lipschitz continuous condition (5) for Newton's method to a type of Hölder continuous condition. Local results were also given in [10–12] for the secant method, King–Werner's method and Müller method, respectively.

In this paper, a local convergence theorem for the secant-like method (3) is established by estimating its convergence radius. Meanwhile, the error estimates matching the convergence order of the method are given.

2. Convergence analysis of the secant-like method (3)

We can state a local convergence theorem for the secant-like method (3).

Theorem 2.1. *Assume that $f \in C^2(D)$. Suppose $x^\star \in D$, $f(x^\star) = 0$, $f'(x^\star) \neq 0$ and $|f'(x^\star)^{-1} f''(x)| \leqslant M$ for some constant $M > 0$. If the initial points x_0, $y_0 \in B(x^\star, r^\star) \subseteq D$ with $r^\star = \frac{s^\star}{M}$, and $s^\star \approx 0.552786405$ is the unique positive root in $\left[0, \frac{4}{5}\right]$ of the following cubic equation*

$$h(s) = 5s^3 - 20s^2 + 24s - 8 = 0, \tag{7}$$

then the sequence $\{x_n\}$ generated by the secant-like method (3) is well defined, $x_n \in B(x^\star, r^\star)$, converges to $x^\star$ with order $p = 1 + \sqrt{2} \approx 2.414$, and the following estimates are satisfied for all $n \geqslant 1$:

$$|x_n - x^\star| \leqslant \frac{r^\star}{C}\left(\frac{C|x_1 - x^\star|}{r^\star}\right)^{F_n}, \tag{8}$$

where $C = \frac{s^\star}{1-s^\star}\sqrt{\frac{s^\star}{4-6s^\star}} \approx 1.111785941$, and $\{F_n\}$ is a generalized Fibonacci sequence defined by $F_1 = F_2 = 1$, and $F_{n+2} = 2F_{n+1} + F_n$ $(n \geqslant 1)$.

Proof. Let $e_n = x_n - x^\star$, and $\Delta_n = y_n - x^\star$ be the errors at the nth iteration for the sequence $\{x_n\}$ and $\{y_n\}$, respectively.

We will show the theorem by induction. Suppose that for some integer $n \geqslant 0$, x_k, and y_k are well defined by (3), and x_k, $y_k \in B(x^\star, r^\star)$ for $k = 0, 1, \ldots, n$. By a simple derivation, we have

$$\begin{aligned} \left|1 - f'(x^\star)^{-1} f[y_n, x_n]\right| &= \left|f'(x^\star)^{-1}\left(f[x^\star, x^\star] - f[y_n, x^\star] + f[y_n, x^\star] - f[y_n, x_n]\right)\right| \\ &= \left|f'(x^\star)^{-1}\left(f[x^\star, y_n, x^\star](-\Delta_n) + f[y_n, x^\star, x_n](-e_n)\right)\right|. \end{aligned} \tag{9}$$

Here, $f[\underbrace{\cdot, \cdot, \ldots, \cdot}_{m+1}]$ are divided differences of order m for $m = 1, 2, \ldots$ (the definitions and properties of them, see [13]). By the properties of divided differences, we have from $|f'(x^\star)^{-1} f''(x)| \leqslant M$ that