Applied Mathematics and Computation 217 (2010) 612–621

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Convergence radius of the modified Newton method for multiple zeros under Hölder continuous derivative

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ARTICLE INFO

Keywords:
Nonlinear equations
Zero-finding
The modified Newton method
Multiple zeros
Convergence radius
Hölder condition
Divided differences

ABSTRACT

In recent years, a lot of iterative methods for finding multiple zeros of nonlinear equations have been presented and analyzed. However, almost all these studies give no information for the convergence radius of the corresponding method. In this paper, we give an estimate of the convergence radius of the well-known modified Newton's method for multiple zeros, when the involved function satisfies a Hölder and center-Hölder continuity condition.

1. Introduction

Finding the zeros of nonlinear equations is one of the most important problems in numerical analysis. In this study, we use iterative methods to find a multiple zero $x^\star$ of multiplicity m ($m > 1$), i.e., $f^{(j)}(x^\star) = 0, j = 0, 1, \ldots, m-1$, and $f^{(m)}(x^\star) \neq 0$, of a nonlinear equation $f(x) = 0$, where $f : D \to \mathbb{R}$ is a smooth function, and D is an open interval.

It is known that, the modified Newton method for multiple zeros is given by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad (1.1)$$

which converges quadratically [1].

There exists a cubically convergent method for multiple zeros, presented by Hansen and Patrick [2]:

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{m+1}{2m}f'(x_n) - \frac{f(x_n)f''(x_n)}{2f'(x_n)}}, \quad (1.2)$$

which is an extension of the classical Halley method of the third-order.

Another cubically convergent method for multiple zeros, is proposed by Traub [3]:

$$x_{n+1} = x_n - \frac{m(3-m)}{2}\frac{f(x_n)}{f'(x_n)} - \frac{m^2}{2}\frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3}, \quad (1.3)$$

which is an extension of the well-known Chebyshev method of the third-order.

In recent years, a lot of methods for multiple zeros have been presented, and many local convergence results have been obtained, see [4–16] and references therein. In general, these results show that if the initial guess x_0 of the corresponding

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doi:10.1016/j.amc.2010.05.098

method is sufficiently close to the zero $x^\star$ of the function involved, then the sequence $\{x_n\}$ generated by this method is well defined, and converges to $x^\star$. But we don't know how close to the zero $x^\star$ the initial guess x_0 should be. That is, these local results give no information on the radius of the convergence ball for the corresponding method.

Here, we say an open ball $U(x^\star, r_\star) \subseteq D$ with center $x^\star$ and radius $r_\star$ is called a convergence ball of an iterative method, if the sequence generated by this iterative method starting from any initial values in it converges. Enlarging the convergence ball of an iterative method is very important, because it shows the extent of difficulty we have to choose iterative initial points. Of course, we will be delighted to have bigger radius for the convergence ball of an iterative method. However, this depends on not only the iterative method we use but also on the conditions of the involved function, such as Lipschitz continuous conditions, Hölder continuous conditions, etc.

It is known that, in the case of a simple zero, many results on the estimate of the radius of the convergence ball have been given for iterative methods, see [17–27]. Wang [17] and Traub [18] gave an exact estimate

$$r^\star = \frac{2}{3K}, \tag{1.4}$$

on the radius of the convergence ball respectively for Newton's method ($m = 1$ in (1.1)), when function f satisfies the Lipschitz continuous condition:

$$|f'(x^\star)^{-1}(f'(x) - f'(y))| \leqslant K|x - y|, \quad \forall x,\ y \in D, \quad \text{for some } K > 0. \tag{1.5}$$

Argyros [20] provided the radius

$$r_\star = \left[\frac{1+p}{K + (1+p)K_0}\right]^{\frac{1}{p}} \tag{1.6}$$

for Newton's method, when function f satisfies the Hölder continuous condition:

$$|f'(x^\star)^{-1}(f'(x) - f'(y))| \leqslant K|x - y|^p, \quad \forall x, y \in D, \quad \text{for some } K > 0 \tag{1.7}$$

and the center-Hölder continuous condition:

$$|f'(x^\star)^{-1}(f'(x) - f'(x^\star))| \leqslant K_0|x - x^\star|^p, \quad \forall x \in D, \quad \text{for some } K_0 > 0, \tag{1.8}$$

where, $0 < p \leqslant 1$. Note that, when $p = 1$ and $K = K_0$, the radius given by (1.6) reduces to that given by (1.4). Otherwise, since $K_0 < K$: $r^\star < r_\star$, and the upper bounds on the distance $|x_n - x^\star| (n \geqslant 0)$ are tighter. This development is very important in computational mathematics, since a wider range of initial guesses x_0 becomes available, and at most as few computations are required to obtain a desired error tolerance. Note also that the radio $\frac{K}{K_0}$ can be arbitrarily large [21]. Moreover, (1.8) is not an additional hypothesis, since in practice the computation of constant K requires that of K_0. Huang [22] generalized the Lipschitz continuous condition (1.5) of Newton's method for another type of Hölder continuous condition. Local results were also given in [23] using even more general conditions for Newton-like methods.

In this paper, we provide the radius

$$r_\star = \left(\frac{\prod_{i=1}^{m}(m+p+1-i)}{(m-1)!(K + (m+p)K_0)}\right)^{\frac{1}{p}} \tag{1.9}$$

for the modified Newton's method (1.1) in the case of multiple zeros, when function f satisfies the Hölder continuous condition:

$$|f^{(m)}(x^\star)^{-1}(f^{(m)}(x) - f^{(m)}(y))| \leqslant K|x - y|^p, \quad \forall x, y \in D, \quad \text{for some } K > 0 \tag{1.10}$$

and the center-Hölder continuous condition:

$$|f^{(m)}(x^\star)^{-1}(f^{(m)}(x) - f^{(m)}(x^\star))| \leqslant K_0|x - x^\star|^p, \quad \forall x \in D, \quad \text{for some } K_0 > 0, \tag{1.11}$$

where, $0 < p \leqslant 1$. Note that, conditions (1.10) and (1.11) are natural generalizations of conditions (1.7) and (1.8) from the case of simple zero ($m = 1$) to multiple zero ($m > 1$), and the radius given by (1.9) reduces to the radius given by (1.6) when $m = 1$. We also provide the error analysis, which matches the convergence order of the modified Newton's method (1.1). The advantages of our approach for $m > 1$ are the same as in the case when $m = 1$, which has been already explained above.

2. Preliminaries

We need the definitions of divided differences, and their properties.

Definition 2.1 [28]. The divided differences $f[a_0, a_1, \ldots, a_k]$, on $k + 1$ distinct points $a_0, a_1, \ldots, a_k$ of a function $f(x)$ are defined by