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A note on Chrystal's equation

P.M. Jordan

Code 7181, Naval Research Laboratory, Stennis Space Ctr., MS 39529–5004, USA

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ABSTRACT

The exact, explicit form of the transcendental solution of Chrystal's equation, a first order nonlinear ordinary differential equation (ODE) of degree two, is derived in terms of the Lambert W-function. It is shown that this case of the general solution is dual-valued over a finite interval and that, for a special case of the coefficients, its zeros involve the Golden ratio. Additionally, a number of applications involving special cases of this ODE are noted and the main properties of the Lambert W-function are briefly reviewed.

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The nonlinear ODE:

$$p^2 + Axp + By + Cx^2 = 0, \tag{1}$$

where A, B, and C are constants and $p := d\,y/d\,x$, has come to be known as Chrystal's equation [1–3]. Special cases of Eq. (1) arise in diverse fields of science and engineering; e.g., the design of parabolic reflectors used in optics and acoustics [4], the study of poroacoustic traveling wave phenomena [5], and the modeling of plasma behavior in fusion physics [6].

The usual approach in treating Eq. (1) is to first solve for p, which yields[1]:

$$p = \frac{1}{2}\left[-Ax \mp \sqrt{(A^2 - 4C)x^2 - 4By}\right], \tag{2}$$

and then make use of the transformation:

$$4By = (A^2 - 4C - z^2)x^2, \tag{3}$$

where $B \neq 0$ is henceforth assumed. If we further assume that $x \neq 0$ and $z \neq a, b$, then the transformed ODE is easily separated and re-expressed as:

$$\frac{-z\,dz}{(z-a)(z-b)} = \frac{dx}{x}, \tag{4}$$

where:

$$a = \frac{1}{2}(Q \pm B), \quad b = \frac{1}{2}(-Q \pm B) \quad \text{and} \quad Q = (2A + B)^2 - 16C. \tag{5}$$

Since the case $a \neq b$, which results in an algebraic solution, has already been examined by a number of other authors, in particular, Ince [1] and Davis [2], it will not be discussed here. Instead, the present Note is devoted to the case $a = b$, and the transcendental solutions which stem from it.

E-mail address: pjordan@nrlssc.navy.mil

[1] It must be noted that Ref. [2, Eq. (20)] contains the following two misprints: The '±' signs and the coefficient 'B' should be replaced with '∓' and '$4B$', respectively.

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To this end, we set the discriminant $Q = 0$, i.e., we assume $(2A + B)^2 = 16C$, from which it follows that $a = b = \pm\frac{1}{2}B$. Consequently, Eq. (4) reduces to:

$$\frac{-z dz}{\left(z \mp \frac{1}{2}B\right)^2} = \frac{dx}{x}, \tag{6}$$

which can be integrated to yield the implicit solution:

$$x\left(z \mp \frac{1}{2}B\right) \exp\left(\frac{\mp B}{2z \mp B}\right) = c, \tag{7}$$

where $c \neq 0$ is an otherwise arbitrary constant. With a little effort, Eq. (7) can be recast as:

$$\pm\frac{x}{\ell} = \left(\frac{\pm B}{2z \mp B}\right) \exp\left(\frac{\pm B}{2z \mp B}\right), \tag{8}$$

where we have set $\ell = 2c/B$ for convenience. Solving now for z yields:

$$z = \pm\frac{1}{2}B\left[1 + \frac{1}{W(\pm x/\ell)}\right], \tag{9}$$

where $W(\cdot)$ denotes the Lambert W-function (see Appendix A). Without loss of generality, we henceforth limit our attention to the '+' (i.e., upper) sign case and substitute Eq. (9) into Eq. (3), thus yielding:

$$y = -\frac{1}{4}\begin{cases} x^2\left[A + \frac{1}{2}B + \frac{B}{2W_0(x/\ell)} + \frac{B}{4W_0^2(x/\ell)}\right], & x\ell > 0; \\ \begin{cases} x^2\left[A + \frac{1}{2}B + \frac{B}{2W_r(-|x/\ell|)} + \frac{B}{4W_r^2(-|x/\ell|)}\right], & |x| \in (0, e^{-1}|\ell|), \\ e^{-2}\ell^2\left(A + \frac{1}{4}B\right), & |x| = e^{-1}|\ell|, \\ \nexists, & |x| > e^{-1}|\ell|, \end{cases} & x\ell < 0; \end{cases} \tag{10}$$

for each $r \in \{-1, 0\}$, where $\nexists$ denotes the fact that y is complex-valued for $|x| > e^{-1}|\ell|$ (see Appendix A) and we have used the fact that $Q = 0$ implies $4C = \frac{1}{4}(2A + B)^2$.

It is noteworthy that if $x\ell < 0$, but $|x| \in (0, e^{-1}|\ell|)$, then there are *two* distinct real values of y for every x, i.e., y is dual-valued on this interval; see Appendix A. What is more, while not defined at $x = 0$, the $\lim_{x\to 0} y(x)$ does in fact exists (i.e., $x = 0$ is a removable singularity of $y(x)$); specifically:

$$\lim_{x\to 0} y(x) = \begin{cases} y_0, & \text{when the branch } W_0 \text{ is considered}, \\ 0, & \text{when the branch } W_{-1} \text{ is considered}, \end{cases} \tag{11}$$

where $y_0 := -\frac{1}{4}c^2/B$ and we have made use of Eq. (A.2).

Observing that the phase space points $(x, y, p) = (0, y_0, p_0)$, where $p_0 := \lim_{(x,y)\to(0,y_0)} p = -\frac{1}{2}|c|$ (see Eq. (2)), and $(x, y, p) = (0, 0, 0)$ satisfy Eq. (1), and introducing the scaled variables $X = x/\ell$ and $Y = c^{-2}By$, we can construct the following piecewise-defined expressions for the transcendental case of the general solution of Eq. (1):

$$Y_0(X) = -\begin{cases} X^2\left[K + \frac{1}{2W_0(X)} + \frac{1}{4W_0^2(X)}\right], & X \in [-e^{-1}, 0) \cup (0, \infty), \\ \frac{1}{4}, & X = 0, \end{cases} \tag{12}$$

$$Y_{-1}(X) = -\begin{cases} X^2\left[K + \frac{1}{2W_{-1}(X)} + \frac{1}{4W_{-1}^2(X)}\right], & X \in [-e^{-1}, 0), \\ 0, & X = 0, \end{cases} \tag{13}$$

where $K = 1/2 + A/B$, and we observe that:

$$Y_0(-e^{-1}) = Y_{-1}(-e^{-1}) = -e^{-2}\left(K - \frac{1}{4}\right). \tag{14}$$

It should be noted that the zero(s) of Y_0 are exactly given by:

$$X_0 = -\frac{1}{2}e^{-1/2} \quad (K = 0), \tag{15}$$

$$X_{1,2} = -\left(\frac{1 \pm \sqrt{1 - 4K}}{4K}\right) \exp\left[-\left(\frac{1 \pm \sqrt{1 - 4K}}{4K}\right)\right] \quad (K \neq 0). \tag{16}$$

Also, it is of interest to note that $X_{1,2}$ are real-valued *only* when $K \leqslant 1/4$, where $X_1 = X_2 = -e^{-1}$ when $K = 1/4$ (see Eq. (14)), and that $X_{1,2}$ are complex-valued, and Y_0 is strictly negative, when $K > 1/4$.