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Some applications of Miller–Mocanu lemma on certain classes of meromorphic functions

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ABSTRACT

In this paper, making use of a linear operator we introduce and study certain new classes of meromorphic functions. We derive some inclusion results. These classes contain many known classes as a special case.

1. Introduction

Let $\sum$ denotes the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the punctured open unit disc $D = \{z: 0 < |z| < 1\}$. Further, let $P_k(\gamma)$ be the class of functions $p(z)$, analytic in $E = D \cup \{0\}$ satisfying $p(0) = 1$ and

$$\int_0^{2\pi} \left|\frac{\operatorname{Re} p(z) - \gamma}{1-\gamma}\right| d\theta \leqslant k\pi, \tag{1.2}$$

where $z = re^{i\theta}$, $k \geqslant 2$, $0 \leqslant \gamma < 1$. This class was introduced by Padmanbhan and Paravatham [9]. For $\gamma = 0$ we obtain the class P_k defined by Pinchuk [11] and $P_2(\gamma) = P(\gamma)$ is the class with positive real part greater than γ.

Also note that for $p \in P_k(\gamma)$ if and only if

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z),$$

where $p_1, p_2 \in P(\gamma)$ for $z \in E$. By $MC(\gamma)$, $MS^*(\gamma)$ and $MK(\gamma)$, we mean the subclasses of meromorphic convex, meromorphic star-like and meromorphic close-to-convex functions of order γ respectively. The class $\sum$ is closed under the convolution or Hadamard product denoted and defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k, \tag{1.3}$$

where

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k.$$

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The study of operators plays a vital role in Geometric Function Theory. In 1999, using the technique of convolution Noor [6] defined an integral operator (see also [7]). Many author's [1–4,10] studied the properties of Noor integral operator and generalize it in many directions. Motivated, from Noor works, in [12] Yuan et al. defined an operator $I_{n,\mu}: \sum \to \sum$ as follows:

$$I_{n,\mu}f(z) = f_{n,\mu}(z) * f(z), \tag{1.4}$$

where

$$f_{n,\mu}(z) * \frac{1}{z(1-z)^{n+1}} = \frac{1}{z(1-z)^{\mu}} \quad n > -1,\ \mu > 0,\ z \in D. \tag{1.5}$$

Using (1.4) and (1.5), one can easily have

$$I_{n,\mu}f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\mu)_{k+1}}{(n+1)_{k+1}} a_k z^k, \tag{1.6}$$

where $(a)_k$ is the Pochhammer symbol given by

$$(a)_0 = 1, \quad (a)_k = a(a+1)(a+2)\cdots(a+k-1), \quad k \in N.$$

From (1.4) and (1.6), it can be easily verified that

$$z(I_{n+1,\mu}f(z))' = (n+1)I_{n,\mu}f(z) - (n+2)I_{n+1,\mu}f(z) \tag{1.7}$$

and

$$z(I_{n,\mu}f(z))' = \mu I_{n,\mu+1}f(z) - (\mu+1)I_{n,\mu}f(z). \tag{1.8}$$

Furthermore, for $c > 0$ the Generalized Bernardi Operator is defined as

$$J_c f(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\,dt. \tag{1.9}$$

Using the operator $I_{n,\mu}$, we define the following new classes of meromorphic functions

Definition 1.1. Let $f \in \sum,\ n > -1,\ \mu > 0,\ 0 \leqslant \gamma < 1,\ z \in D$, then $f \in MV_k(n,\mu,\gamma)$ if and only if

$$-\frac{(z(I_{n,\mu}f(z))')'}{(I_{n,\mu}f(z))'} \in P_k(\gamma).$$

Definition 1.2. Let $f \in \sum,\ n > -1,\ \mu > 0,\ 0 \leqslant \gamma < 1,\ z \in D$, then $f \in MR_k(n,\mu,\gamma)$ if and only if

$$-\frac{z(I_{n,\mu}f(z))'}{I_{n,\mu}f(z)} \in P_k(\gamma).$$

It can be easily observed that

$$f \in MV_k(n,\mu,\gamma) \text{ if and only if } zf' \in MR_k(n,\mu,\gamma).$$

Definition 1.3. Let $f \in \sum,\ n > -1,\ \mu > 0,\ 0 \leqslant \alpha,\ \beta < 1,\ z \in D$, then $f \in MT_k^*(n,\mu,\alpha,\beta)$ if and only if there exists $g \in MR_2(n,\mu,\alpha)$ such that

$$-\frac{z(I_{n,\mu}f(z))'}{I_{n,\mu}g(z)} \in P_k(\beta).$$

Remark 1.1. For special values of parameters n, μ, γ and k, we have many known classes of meromorphic functions, see [8,12].

2. Preliminary results

Lemma 2.1 [5]. *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\Psi(u,v)$ be a complex valued function satisfying the conditions:*

(i) $\Psi(u,v)$ is continuous in $D \subset C^2$,
(ii) $(1,0) \in D$ and $\operatorname{Re}\Psi(1,0) > 0$,
(iii) $\operatorname{Re}\Psi(iu_2, v_1) \leqslant 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$.

If $h(z)$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $\operatorname{Re}\Psi(h(z), zh'(z)) > 0$ for $z \in E$, then $\operatorname{Re} h(z) > 0$ in E.