Applied Mathematics and Computation 217 (2010) 3717–3725

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Approximate solution of the system of nonlinear integral equation by Newton–Kantorovich method

Z.K. Eshkuvatov [a,c,*], Anvarjon Ahmedov [b,c], N.M.A. Nik Long [a,c], O. Shafiq [c]

[a] Department of Mathematics, Faculty of Science, Universiti Putra Malaysia (UPM), Malaysia
[b] Department of Process and Food Engineering, Faculty of Engineering, UPM, Malaysia
[c] Institute for Mathematical Research, UPM, Malaysia

ARTICLE INFO

Keywords:
Newton–Kantorovich method
Rate of convergence
Nonlinear operator
Volterra integral equation
Trapezoidal formula

ABSTRACT

The Newton–Kantorovich method is developed for solving the system of nonlinear integral equations. The existence and uniqueness of the solution are proved, and the rate of convergence of the approximate solution is established. Finally, numerical examples are provided to show the validity and the efficiency of the method presented.

1. Introduction

Solving the nonlinear integral equations (IEs) by linearization method is a popular approach by many researches [1,2,4–6], and the necessity of Volterra IEs was emphasized in Baker [3]. For the system of nonlinear integral equations, Glushkov et al. [7] introduced the models of developing systems for describing a large class of problems in economics, ecology, medicine, and other fields of applied mathematics. Boikov and Tynda [4] developed Newton–Kantorovich method for solving the system of nonlinear integral equations (named it two commodity models) as

$$\left.\begin{aligned} &x(t)-\int_{y(t)}^{t}h(t,\tau)g(\tau)x(\tau)d\tau=0,\\ &\int_{y(t)}^{t}k(t,\tau)[1-g(\tau)]x(\tau)d\tau=f(t),\end{aligned}\right\} \tag{1.1}$$

where $0<t_0\leqslant t\leqslant T,\ y(t)<t$ with given the functions $h(t,\tau),k(t,\tau)\in C_{[t_0,\infty]\times[t_0,\infty]}f(t),\ g(t)\in C_{[t_0,\infty]}\ (0<g(t)<1)$ and the unknown functions $x(t)\in C_{[t_0,\infty]},y(t)\in C^1_{[t_0,\infty]}$, and proved the existence, uniqueness and rate of convergence of the approximate solution for Eq. (1.1).

In this work we further investigate the system of nonlinear integral equations of the form

$$\left.\begin{aligned} &x(t)-\int_{y(t)}^{t}h(t,\tau)x^2(\tau)d\tau=0,\\ &\int_{y(t)}^{t}k(t,\tau)x^2(\tau)d\tau=f(t),\end{aligned}\right\} \tag{1.2}$$

where $x(t)$ and $y(t)$ are unknown functions defined on $[t_0,\infty)$, $t_0>0$, and $h(t,\tau),k(t,\tau)\in C_{[t_0,\infty]\times[t_0,\infty]},\ f(t)\in C_{[t_0,\infty]}$ are given functions.

Stable computational scheme is given in detail in Section 2. The existence and uniqueness of the solution and rate of convergence of the approximate solution are proved in Section 3. Discretization of the method is described in Section 4 and in Section 5, numerical examples are given. Finally we end up the theoretical findings and experimental works with conclusions in Section 6.

* Corresponding author at: Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, UPM, Malaysia.
E-mail addresses: ezaini@science.upm.edu.my (Z.K. Eshkuvatov), nmasri@math.upm.edu.my (N.M.A. Nik Long).

0096-3003/$ - see front matter Crown Copyright © 2010 Published by Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2010.09.068

2. Description of the method

To find the unknown functions $x(t)$ and $y(t)$ in Eq. (1.4), we introduce the notations

$$\left.\begin{array}{l} P_1(x(t),y(t)) = x(t) - \int_{y(t)}^{t} h(t,\tau)x^2(\tau)d\tau = 0, \\ P_2(x(t),y(t)) = f(t) - \int_{y(t)}^{t} k(t,\tau)x^2(\tau)d\tau = 0, \end{array}\right\}$$

where $0 < t_0 \leqslant t \leqslant T$, then on the interval $[t_0,T]$ the system of Eq. (1.4) can be reduced to the operator equation

$$P(X) = (P_1(X), P_2(X)) = (0,0), \tag{2.1}$$

where X denotes a vector function $X = (x(t),y(t))$. We solve (2.1) by the modified Newton–Kantorovich method, to do this end we write the approximate equation

$$P'(X_0)(X - X_0) + P(X_0) = 0, \quad X_0 = (x_0(t), y_0(t)). \tag{2.2}$$

It is known that the derivative $P'(X_0)$ of the nonlinear operator $P(X)$ at the point X_0 is determined by the matrix:

$$P'(X_0) = \begin{pmatrix} \frac{\partial P_1}{\partial x}\Big|_{(x_0,y_0)} & \frac{\partial P_1}{\partial y}\Big|_{(x_0,y_0)} \\ \frac{\partial P_2}{\partial x}\Big|_{(x_0,y_0)} & \frac{\partial P_2}{\partial y}\Big|_{(x_0,y_0)} \end{pmatrix}.$$

Consequently Eq. (2.2) has the form

$$\left.\begin{array}{l} \frac{\partial P_1}{\partial x}\Big|_{(x_0,y_0)}(\Delta x(t)) + \frac{\partial P_1}{\partial y}\Big|_{(x_0,y_0)}(\Delta y(t)) = -P_1(x_0(t),y_0(t)), \\ \frac{\partial P_2}{\partial x}\Big|_{(x_0,y_0)}(\Delta x(t)) + \frac{\partial P_2}{\partial y}\Big|_{(x_0,y_0)}(\Delta y(t)) = -P_2(x_0(t),y_0(t)), \end{array}\right\} \tag{2.3}$$

where $\Delta x(t) = x_1(t) - x_0(t)$, $\Delta y(t) = y_1(t) - y_0(t)$. With the given initial point $X_0 = (x_0(t),y_0(t))$, we evaluate $P'(X_0)$ by the definition

$$\begin{aligned} \frac{dP_1}{dx}\Big|_{(x_0,y_0)} &= \lim_{s\to 0} \frac{P_1(x_0 + sx, y_0) - P_1(x_0,y_0)}{s} \\ &= \lim_{s\to 0} \frac{1}{s}\left[x_0(t) + sx(t) - \int_{y_0(t)}^{t} h(t,\tau)[x_0(\tau) + sx(\tau)]^2 d\tau - x_0(t) + \int_{y_0(t)}^{t} h(t,\tau)x_0^2(\tau)d\tau\right] \\ &= x(t) - 2\int_{y_0(t)}^{t} h(t,\tau)x_0(\tau)x(\tau)d\tau, \end{aligned}$$

$$\frac{dP_1}{dy}\Big|_{(x_0,y_0)} = \lim_{s\to 0} \frac{P_1(x_0, y_0 + sy) - P_1(x_0,y_0)}{s} = \lim_{s\to 0} \frac{\int_{y_0(t)}^{y_0(t)+sy(t)} h(t,\tau)x_0^2(\tau)d\tau}{s} = h(t,y_0(t))x_0^2(y_0(t))y(t).$$

In a similar way, we obtain

$$\begin{aligned} \frac{dP_2}{dx}\Big|_{(x_0,y_0)} &= -2\int_{y_0(t)}^{t} k(t,\tau)x_0(\tau)x(\tau)d\tau, \\ \frac{dP_2}{dy}\Big|_{(x_0,y_0)} &= k(t,y_0(t))x_0^2(y_0(t))y(t). \end{aligned}$$

Therefore the system (2.3) becomes

$$\begin{aligned} &\Delta x(t) - 2\int_{y_0(t)}^{t} h(t,\tau)x_0(\tau)\Delta x(\tau)d\tau + h(t,y_0(t))x_0^2(y_0(t))\Delta y(t) \\ &\quad = -x_0(t) + \int_{y_0(t)}^{t} h(t,\tau)x_0^2(\tau)d\tau, \\ &\quad\; -2\int_{y_0(t)}^{t} k(t,\tau)x_0(\tau)\Delta x(\tau)d\tau + k(t,y_0(t))x_0^2(y_0(t))\Delta y(t) \\ &\quad = -f(t) + \int_{y_0(t)}^{t} k(t,\tau)x_0^2(\tau)d\tau, \end{aligned} \tag{2.4}$$

Solving Eq. (2.4) for $\Delta x(t)$ and $\Delta y(t)$ gives $(x_1(t),y_1(t))$. By continuing this process, we obtain a sequence of approximate solutions $(x_m(t),y_m(t))$ found from the following system