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Homogeneous polynomials as Lyapunov functions in the stability research of solutions of difference equations

Oleksiy A. Ignatyev

Department of Statistics and Probability, Michigan State University, A408 Wells Hall, East Lansing, MI 48824-1027, USA

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ABSTRACT

The linear autonomous system of difference equations $x(n+1) = Ax(n)$ is considered, where $x \in \mathbb{R}^k$, A is a real nonsingular $k \times k$ matrix. In this paper it has been proved that if $W(x)$ is any homogeneous polynomial of m-th degree in x, then there exists a unique homogeneous polynomial $V(x)$ of m-th degree such that $\Delta V = V(Ax) - V(x) = W(x)$ if and only if $\lambda_1^{i_1}, \lambda_2^{i_2}, \ldots, \lambda_k^{i_k} \neq 1$ $(i_1 + i_2 + \cdots + i_k = m,\ i_j \geqslant 0)$ where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the eigenvalues of the matrix A. The theorem on the instability has also been proved.

1. Introduction and preliminaries

The theory of discrete dynamical systems has grown tremendously in the last decade. Difference equations can arise in a number of ways. They may be the natural model of a discrete process (in combinatoric, for example) or they may be a discrete approximation of a continuous process. The growth of the theory of difference systems has been strongly promoted by the advanced technology in scientific computation and the large number of applications to models in biology, engineering, and other physical sciences. For example, in papers [1,2] systems of difference equations are applied as natural models of populations dynamics, in [3] difference equations are applied as a mathematical model in genetics.

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbations which duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. Thus impulsive differential equations, that is, differential equations involving impulse effects, appear as a natural description of observed evolution phenomena of several real world problems [4–6]. The early work on differential equations with impulse effect were summarized in monograph [6] in which the foundations of this theory were described. In recent years, the study of impulsive systems has received an increasing interest [7–13]. In fact, an impulsive system consists of a continuous system which is governed by ordinary differential equations and a discrete system which is governed by difference equations. So the dynamics of impulsive systems essentially depends on properties of the corresponding difference systems, and this confirms the importance of studying the qualitative properties of difference systems.

It is the stability and asymptotic behaviour of solutions of these models that is especially important to many investigators. The stability of a discrete process is the ability of the process to resist *a priori* unknown small influences. A process is said to be stable if such disturbances do not change it. This property turns out to be of utmost importance since, in general, an individual predictable process can be physically realized only if it is stable in the corresponding natural sense. One of the

E-mail address: ignatyev@stt.msu.edu

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most powerful method used in stability theory is Lyapunov's direct method [14]. This method consists in the use of an auxiliary function (the Lyapunov function).

Consider the system of difference equations

$$x(n+1) = f(n, x(n)), \quad f(n, 0) = 0, \tag{1.1}$$

where $n = 0, 1, 2, \dots$ is discrete time, $x(n) = (x_1(n), \dots, x_k(n))^T \in \mathbb{R}^k,\ f = (f_1, \dots, f_k)^T \in \mathbb{R}^k$. System (1.1) admits the trivial solution

$$x(n) = 0. \tag{1.2}$$

Denote $x(n, n_0, x^0)$ the solution of system (1.1) coinciding with $x^0 = (x_1^0, x_2^0, \dots, x_k^0)^T$ under $n = n_0$. We also denote $\mathbb{Z}_+$ the set of nonnegative real integers, $\mathbb{N}_{n_0} = \{n \in \mathbb{Z}_+ : n \geqslant n_0\},\ \mathbb{N} = \{n \in \mathbb{Z}_+ : n \geqslant 1\},\ B_r = \{x \in \mathbb{R}^k : \|x\| \leqslant r\}$.

By analogy to ordinary differential equations, let us introduce the following definitions.

Definition 1.1. The trivial solution of system (1.1) is said to be stable if for any $\varepsilon > 0,\ n_0 \in \mathbb{Z}_+$ there exists a $\delta = \delta(\varepsilon, n_0) > 0$ such that $\|x^0\| < \delta$ implies $\|x(n, n_0, x^0)\| < \varepsilon$ for $n \in \mathbb{N}_{n_0}$. Otherwise the trivial solution of system (1.1) is called unstable. If in this definition δ can be chosen independent of n_0 (i.e. $\delta = \delta(\varepsilon)$), then the zero solution of system (1.1) is said to be uniformly stable.

Definition 1.2. Solution (1.2) of system (1.1) is said to be attractive if for any $n_0 \in \mathbb{Z}_+$ there exists an $\eta = \eta(n_0) > 0$ and for any $\varepsilon > 0$ and $x^0 \in B_\eta$ there exists a $\sigma = \sigma(\varepsilon, n_0, x^0) \in \mathbb{N}$ such that $\|x(n, n_0, x^0)\| < \varepsilon$ for all $n \in \mathbb{N}_{n_0+\sigma}$.

In other words, solution (1.2) of system (1.1) is called attractive if

$$\lim_{n\to\infty} \|x(n, n_0, x^0)\| = 0. \tag{1.3}$$

Definition 1.3. The trivial solution of system (1.1) is said to be uniformly attractive if for some $\eta > 0$ and for each $\varepsilon > 0$ there exists a $\sigma = \sigma(\varepsilon) \in \mathbb{N}$ such that $\|x(n, n_0, x^0)\| < \varepsilon$ for all $n_0 \in \mathbb{Z}_+,\ x^0 \in B_\eta$ and $n \geqslant n_0 + \sigma$.

In other words, solution (1.2) of system (1.1) is called uniformly attractive if (1.3) holds uniformly in $n_0 \in \mathbb{Z}_+,\ x^0 \in B_\eta$.

Definition 1.4. The zero solution of system (1.1) is called:

- asymptotically stable if it is stable and attractive;
- uniformly asymptotically stable if it is uniformly stable and uniformly attractive.

Definition 1.5. The trivial solution of system (1.1) is said to be exponentially stable if there exist $M > 0$ and $\eta \in (0, 1)$ such that $\|x(n, n_0, x^0)\| < M\|x^0\|\eta^{n-n_0}$ for $n \in \mathbb{N}_{n_0}$.

A great number of papers is devoted to investigation of the stability of solution (1.2) of system (1.1). The general theory of difference equations and the base of the stability theory are stated in [15–18]. It has been proved in [19] that if system (1.1) is autonomous (i.e. f does not depend explicitly in n) or periodic (i.e. there exists $\omega \in \mathbb{N}$ such that $f(n, x) \equiv f(n + \omega, x)$), then from the stability of solution (1.2) it follows its uniform stability, and from its asymptotic stability it follows its uniform asymptotic stability. Papers [20–22] deal with the stability investigation of the zero solution of system (1.1) when this system is periodic or almost periodic.

Let us formulate the main theorems of Lyapunov's direct method about the stability of the zero solution of the system of autonomous difference equations

$$x(n+1) = f(x(n)). \tag{1.4}$$

These statements have been mentioned in [16, Theorems 4.20 and 4.27]. They are connected with the existence of an auxiliary function $V(x)$; the analog of its derivative is the variation of V relative to (1.4) which is defined as $\Delta V(x) = V(f(x)) - V(x)$.

Theorem 1.6. *If there exists a positive definite continuous function $V(x)$ such that $\Delta V(x)$ relative to (1.4) is negative semi-definite function or identically equals to zero, then the trivial solution of system (1.4) is stable.*

Theorem 1.7. *If there exists a positive definite continuous function $V(x)$ such that $\Delta V(x)$ relative to (1.4) is negative definite, then the trivial solution of system (1.4) is asymptotically stable.*

Theorem 1.8. *If there exists a continuous function $V(x)$ such that $\Delta V(x)$ relative to (1.4) is negative definite, and the function V is not positive semi-definite, then the trivial solution of system (1.4) is unstable.*

Consider the autonomous system

$$x(n+1) = Ax(n) + X(x(n)), \tag{1.5}$$