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The Gerber–Shiu discounted penalty function in the risk process with phase-type interclaim times

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ABSTRACT

In this paper, we consider the Gerber–Shiu discounted penalty function for the Sparre Anderson risk process in which the interclaim times have a phase-type distribution. By the Markov property of a joint process composed of the risk process and the underlying Markov process, we provide a new approach to prove the systems of integro-differential equations for the Gerber–Shiu functions. Closed form expressions for the Gerber–Shiu functions are obtained when the claim amount distribution is from the rational family. Finally we compute several numerical examples intended to illustrate the main results.

1. Introduction

Let $(\Omega, \mathscr{F}, \mathrm{P})$ be a completed probability space containing all the variables defined in the paper. Consider a continuous time Sparre Anderson (renewal) risk process

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Z_i, \quad t \geqslant 0, \tag{1.1}$$

where $u \geqslant 0$ is the initial capital, $c > 0$ is the constant rate of premium. $\{N(t), t \geqslant 0\}$ is an ordinary renewal process counting the number of claims up to time t, $\{Z_i, i \geqslant 1\}$ are independent non-negative claim-size random variables with common distribution P and density p. Let $V_i, i = 1, 2, \ldots$ be the interclaim time random variables. They are assumed to be independent and have common distribution function K, density function k, and Laplace transforms $\hat{k}(s) = \int_0^\infty e^{-sx} k(x) dx$. Further, we assume that $cE(V_i) > E(Z_i)$, providing a positive safety loading factor.

Let $T = \inf\{t : U(t) \leqslant 0\}$ (with $\inf\{\emptyset\} = \infty$) be the time of ruin, $|U(T)|$ be the deficit at ruin, and $U(T-)$ the surplus immediately before ruin. For $\delta \geqslant 0$, the Gerber–Shiu discounted penalty function (Gerber–Shiu function for simplicity) is defined by

$$\phi(u) = E[e^{-\delta T} I(T < \infty) w(U(T-), |U(T)|) | U(0) = u], \tag{1.2}$$

where $w(s, t)$ is a non-negative penalty function of the surplus prior to ruin and the deficit at ruin, I is an indicator function.

The evaluation of the Gerber–Shiu discounted penalty function, first introduced in Gerber and Shiu [8], is now one of the main research problems in ruin theory. See, for example, Cai and Dickson [3] for the compound Poisson risk process with

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interest, Li and Garrido [14] for the Erlang (n) risk process, Gerber and Shiu [9] for the generalized Erlang (n) (i.e., the distribution is the convolution of n exponential distribution, with possibly different parameters) risk process, Cai [4] for the compound Poisson risk process with debit interest, Willmot [22] and Landriault and Willmot [12] for the renewal risk model with special claim-sizes and the references therein. Many ruin-related quantities can be analyzed by appropriately choosing special penalty function w, for example, let $\delta = 0$ and $w(s,t)$ be the Dirac function with respect to $s = x, t = y, \phi(u)$ becomes the joint defective density function of surplus prior to ruin and the deficit at ruin, denoted by $f(x,y|u)$.

The phase-type distribution is one of the most commonly distributions in queuing theory and risk theory. Recently, more and more attention has been paid to the Sparre Andersen model in which the interclaim times are phase-type distributed. See, for example, Albrecher and Boxma [1] analyze the discounted penalty function by means of Laplace–Stieltjes transforms, Dickson and Drekic [6] and Pitts and Politis [17] consider the joint density of the surplus before and after ruin. Li [13], consider the time of recovery and the maximum severity of ruin.

In this paper, we consider a Sparre Anderson risk process with phase-type interclaim times and derive explicit formulas for the penalty functions for arbitrary penalty function w. The rest paper is organized as follows. Section 2 derives a system of integro-differential equations for the Gerber–Shiu functions by the Markov property of a joint process composed of the risk process and the underlying Markov process generated from the phase-type interclaim times. Section 3 fully analyzes the Gerber–Shiu functions. Section 4 contains several numerical examples to illustrate the main results.

2. A set of Integro-differential equations

In the remainder of this paper, we assume that the distribution of the interclaim time random variable K is phase-type with representation $(\boldsymbol{\alpha}, \mathbf{B}, \mathbf{b})$, where $\boldsymbol{\alpha}$ and $\mathbf{b}$ are row vectors of length n and $\mathbf{B}$ is a $n \times n$ matrix. In particular, $\mathbf{b}^{\mathrm{T}} = -\mathbf{B}\mathbf{e}^{\mathrm{T}}$, where $\mathbf{e}$ denote a row vector of length n with all elements being one. By the definition of phase-type distributions, each of the interclaim times $V_k, k = 1,2,\ldots$ corresponds to the time to absorption in a terminating continuous-time Markov Chain, say, $J_t^{(k)}$ with n transient states $\{\mathscr{E}_1, \mathscr{E}_2, \ldots \mathscr{E}_n\}$ and one absorbing state $\mathscr{E}_0$. Following Asmussen [2] and Rolski et al. [20],

$$K(t) = 1 - \boldsymbol{\alpha} e^{t\mathbf{B}} \mathbf{e}^{\mathrm{T}}, \quad t \geqslant 0,$$

$$k(t) = \boldsymbol{\alpha} e^{t\mathbf{B}} \mathbf{b}^{\mathrm{T}}, \quad t \geqslant 0,$$

and

$$\hat{k}(s) = \int_0^\infty e^{-st} k(t) dt = \boldsymbol{\alpha}(s\mathbf{I} - \mathbf{B})^{-1} \mathbf{b}^{\mathrm{T}}. \tag{2.1}$$

For $i = 1,2,\ldots,n$, let $\phi(u;i)$ denote the Gerber–Shiu function given $U(0) = u$ and $J_0^{(1)} = \mathscr{E}_i$, that is,

$$\phi(u;i) = E[e^{-\delta T} I(T < \infty) w(U(T-), |U(T)|) | U(0) = u, J_0^{(1)} = \mathscr{E}_i], \quad i = 1,2,\ldots,n.$$

Then the Gerber–Shiu function may be computed by

$$\phi(u) = \boldsymbol{\alpha} \boldsymbol{\phi}(u)$$

where $\boldsymbol{\phi}(u) = (\phi(u;1),\ldots,\phi(u;n))^{\mathrm{T}}$ is a column vector of functions.

Our first result gives integro-differential equations for Gerber–Shiu functions. Gerber and Shiu [9] prove the integro-differential equations for Gerber–Shiu functions in the generalized Erlang (n) risk model by using the memoryless property of the exponential distribution. Schmidli [21] extends the results of Gerber and Shiu [9] to the case where the interclaim times have a phase-type distribution and shows Eq. (2.2) (see below) and Ko [10] provide a proof of Eq. (2.2). Motivated by these references, the following theorem presents an alternative approach from the theory of Markov process to prove the equations for Gerber–Shiu functions.

Theorem 2.1. *Let $u \geqslant 0$, the vector $\boldsymbol{\phi}(u)$ satisfies*

$$c\boldsymbol{\phi}'(u) + (\mathbf{B} - \delta \mathbf{I})\boldsymbol{\phi}(u) + \left[\int_0^u \boldsymbol{\alpha}\boldsymbol{\phi}(u-x)p(x)dx + \omega(u)\mathbf{I}\right]\mathbf{b}^{\mathrm{T}} = \mathbf{0}, \tag{2.2}$$

where $\omega(u) = \int_u^\infty w(u, x-u)p(x)dx, \mathbf{I} = \mathrm{diag}(1,1,\ldots,1), \mathbf{0}$ denotes a column vector of length n with all elements being 0.

Proof. We first construct a Markov process $\{J(t), t \geqslant 0\}$ by piecing the $\left\{J_t^{(k)}\right\}$ together,

$$J(t) = \left\{J_t^{(1)}\right\}, 0 \leqslant t < V_1, \quad J(t) = \left\{J_{t-V_1}^{(2)}\right\}, \quad V_1 \leqslant t < V_1 + V_2, \ldots$$

Then as in Jacobson [11], the joint process $\{(U(t), J(t)), t \geqslant 0\}$ is a Markovian additive process. To prove (2.2), consider a short time interval $[0,h]$. consider a short time interval $[0,h]$, during which three possible events may occur:

(1) no transitions of states occur during time interval $[0,h]$;
(2) a transitions of states occur during time interval $[0,h]$, but no claim occurs;
(3) at least one claim occurs during time interval $[0,h]$,